2019 57th Annual Allerton Conference on Communication, Control, and Computing (Allerton)
Allerton Park and Retreat Center
Monticello, IL, USA, September 24-27, 2019

# From Alignment to Acyclic Chains: Lexicographic Performance Bounds for Index Coding

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***Abstract*—In this work, we study the information-theoretic performance bounds for the index coding problem. Recently, *weighted* alignment chain models were proposed by generalizing the alignment chain model of Maleki et al. This led to derivation of single-letter performance bounds that are strictly tighter than both the well-known maximum acyclic induced subgraph (MAIS) bound and the internal conflict bound. Here, we propose a more general chain model, namely the acyclic chain. Unlike weighted alignment chains that are constructed by individual messages, acyclic chains can deal with set of messages as their components. This allows a recursive development of new performance bounds. The new acyclic chain bounds subsume the weighted alignment chain bounds and can be strictly tighter. Moreover, a key drawback of the weighted alignment chain bounds is that their improvement over the MAIS bound is limited to a fixed constant value that does not scale with the problem size. In contrast, we show that the new acyclic chain bounds have a desired multiplicative property under the lexicographic product of the index coding side information graphs. As such, the gap between these new bounds and the MAIS bound is not fixed, but can be magnified to a multiplicative factor which is polynomial in the problem size.**

## I. Introduction

*Index coding* [1] studies the problem of broadcasting $n$ messages from a server to multiple receivers, where each receiver requests a unique message and has prior knowledge of some other messages as side information. The objective of index coding is to minimize the codeword length to be broadcast from the server through exploiting the side information at the receivers. Index coding has a simple, yet fundamental model with deep connections to other important problems such as network coding, distributed storage, and coded caching. See [2] and the references therein.

In this paper we focus on performance bounds (information-theoretic converse results) for index coding. Several bounds exist in the literature, including the maximum acyclic induced subgraph (MAIS) bound [3], the internal conflict bound based on the *alignment chain* model [4], [5], and the polymatroidal (PM) bound [6], [7]. The MAIS bound and the internal conflict bound can outperform each other, while they are both subsumed by the PM bound. It has been shown in [8] that the PM bound, incorporating all the constraints stipulated by the receiver decoding requirements, is the tightest bound that can be obtained by Shannon-type inequalities. However, the computational complexity of the PM bound is much higher than that of the MAIS bound or the internal conflict bound, and can be forbidding for problems with a large number of messages.

Therefore, it is of interest to find bounds that are tighter than the MAIS bound and the internal conflict bound, but not as computationally intensive as the PM bound. In [9], the key components of the MAIS bound and the internal conflict bound were combined together to develop the *weighted* alignment chain model – a chain constituted by a number of concatenated acyclic structures. The weighted alignment chain model was then used to derive two performance bounds: the *singleton* and the *disjoint* weighted alignment chain bounds. Both bounds subsume both the MAIS and internal conflict bound. However, as we show in this paper, a key drawback of the weighted alignment chain bounds in [9] is that their improvement over the MAIS bound is limited to a constant and does not scale when the problem size grows.

In this paper, we further explore the potential of concatenated acyclic structures in providing tighter converse results. That is, we build the chain based on *sets* of messages rather than just *individual* messages. More specifically, we build structures constituted by groups of messages that are *acyclic at the set level*, and build the chain as a concatenation of such structures. Such generalization leads to a recursive development of new performance bounds, which include the weighted alignment chain bounds as special cases. Our proposed chains, namely, *singleton acyclic chains* and *disjoint acyclic chains*, as well as their corresponding performance bounds are presented in Section IV. In Section V, we establish an important structural property for the proposed acyclic chain bounds under the lexicographic product of index coding side information graphs, which indicates that the gap between these bounds and the MAIS bound can be magnified to a multiplicative factor that is polynomial in the problem size.

*Notation:* For non-negative integers $a$ and $b$, $[a]$ denotes the set $\{1, 2, \cdots, a\}$, and $[a:b]$ denotes the set $\{a, a+1, \cdots, b\}$. If $a > b$, $[a:b] = \emptyset$. For a set $S$, $|S|$ denotes its cardinality.

## II. System Model and Preliminaries

Assume that there are $n$ messages, $x_i \in \{0,1\}^{u_i}, i \in [n]$, where $u_i$ is the length of binary message $x_i$. For brevity, when we say message $i$, we mean message $x_i$. Let $X_i$ be the random variable corresponding to $x_i$. We assume that $X_1, \ldots, X_n$ are independent and uniformly distributed. For any $S \subseteq [n]$, set $S^c \doteq [n] \setminus S$, $\mathbf{x}_S \doteq (x_i, i \in S)$, and $\mathbf{X}_S \doteq (X_i, i \in S)$. By convention, $\mathbf{x}_\emptyset = \mathbf{X}_\emptyset = \emptyset$.

There is a single server that contains all messages $\mathbf{x}_{[n]}$ and is connected to all receivers via a noiseless broadcast link of normalized capacity $C = 1$. Let $y$ be the output of the server, which is a function of $\mathbf{x}_{[n]}$. There are $n$ receivers, where receiver $i \in [n]$ wishes to obtain $x_i$ and knows $\mathbf{x}_{A_i}$ as side information for some $A_i \subseteq [n] \setminus \{i\}$. The set of indices of *interfering messages* at receiver $i$ is denoted by the set $B_i = (A_i \cup \{i\})^c$. For any set $K \subseteq [n]$, define $B_K$ as

$$B_K \doteq \bigcap_{i \in K} B_i, \tag{1}$$

denoting the *common* interfering message set of the receivers in $K$. It can be verified that the following property holds:

$$B_K \subseteq B_{K'}, \qquad \forall K' \subseteq K \subseteq [n]. \tag{2}$$

We define a $(\mathbf{u}, r) = ((u_i, i \in [n]), r)$ *index code* by

- An encoder $\phi : \prod_{i \in [n]} \{0,1\}^{u_i} \to \{0,1\}^r$, which maps the messages $\mathbf{x}_{[n]}$ to an $r$-bit sequence $y$, and
- $n$ decoders, one for each receiver $i \in [n]$, such that $\psi_i : \{0,1\}^r \times \prod_{k \in A_i} \{0,1\}^{u_k} \to \{0,1\}^{u_i}$ maps the received $y$ and the side information $\mathbf{x}_{A_i}$ to $\hat{x}_i$.

We say that a rate tuple $\mathbf{R} = (R_i, i \in [n])$ is achievable if for every $\epsilon > 0$, there exist a $(\mathbf{u}, r)$ index code such that

$$R_i = \frac{u_i}{r}, \quad i \in [n], \tag{3}$$

and that $\mathsf{P}\{(\hat{X}_1, \ldots, \hat{X}_n) \neq (X_1, \ldots, X_n)\} \leq \epsilon$. The capacity region $\mathscr{C}$ of a problem is the closure of the set of all its achievable $\mathbf{R}$. The symmetric capacity is defined as

$$C_{\text{sym}} = \max\{R : (R, \cdots, R) \in \mathscr{C}\}. \tag{4}$$

And the broadcast rate $\beta$, which characterizes the minimum number of the required transmission from the server to satisfy all the receivers when the messages are of same length, is defined as the reciprocal of the symmetric capacity,

$$\beta = 1/C_{\text{sym}}. \tag{5}$$

Any index coding problem can be represented by a sequence $(i|j \in A_i), i \in [n]$. For example, for $A_1 = \emptyset$, $A_2 = \{3\}$, and $A_3 = \{2\}$, we write $(1|-)$, $(2|3)$, $(3|2)$. It can also be represented by a side information graph $\mathcal{G}$ with $n$ vertices, in which vertex $i$ represents message $i$, and a directed edge $(i, j)$ means that $i \in A_j$. Hence, in the rest of the paper, the index coding problem with $n$ messages for which receiver $i$ knows side information $A_i$ is compactly referred to as the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, or sometimes simply the index coding problem $\mathcal{G}$. We also denote the broadcast rate of the problem $\mathcal{G}$ as $\beta(\mathcal{G})$, when such dependence is to be emphasized.

For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, for any nonempty set $S \subseteq [n]$, $\mathcal{G}|_S$ denotes the subgraph/subproblem of $\mathcal{G}$ induced by $S$. If $\mathcal{G}|_S$ is acyclic, we simply say that the set $S$ forms an acyclic structure or $S$ is acyclic.

For any directed graph $\mathcal{G}$, use $V(\mathcal{G})$ to denote the vertex set of the graph. Hence, for the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, $V(\mathcal{G}) = [n]$. Let $E(\mathcal{G}) = \{(i, j) : i, j \in V(\mathcal{G})$ and there is a directed edge from $i$ to $j\}$ be the set of edges of $\mathcal{G}$. The lexicographic product of directed graphs is stated as follows.

*Definition 1 (Lexicographic Product, [2], [10]):* For any two index coding problems $\mathcal{G}_0$ and $\mathcal{G}_1$, the problem

$$\mathcal{G} = \mathcal{G}_0 \circ \mathcal{G}_1,$$

if and only if $V(\mathcal{G}) = V(\mathcal{G}_0) \times V(\mathcal{G}_1) = \{(i_1, i_2) : i_1 \in V(\mathcal{G}_1), i_2 \in V(\mathcal{G}_2)\}$, and $E(\mathcal{G}) = \{((i_1, i_2), (j_1, j_2)) : (i_1, j_1) \in E(\mathcal{G}_0)$ or $i_1 = j_1, (i_2, j_2) \in E(\mathcal{G}_1)\}$.

We briefly review the MAIS bound [3] below.

*Proposition 1 (MAIS bound, [3]):* For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, its $\beta(\mathcal{G})$ is lower bounded as

$$\beta(\mathcal{G}) \geq \beta_{\text{MAIS}}(\mathcal{G}) \doteq \max_{S \subseteq [n]: \mathcal{G}|_S \text{ is acyclic}} |S|.$$

As extensions to the orignal alignment chain model [4], the singleton and disjoint weighted alignment chain models were developed in [9]. For a given problem $\mathcal{G}$, we use $\beta_{\text{SW}}(\mathcal{G})$ and $\beta_{\text{DW}}(\mathcal{G})$ to denote the singleton and disjoint weighted alignment chain bounds, respectively. The singleton weighted alignment chain is a special case of the disjoint weighted alignment chain, and thus $\beta_{\text{SW}}(\mathcal{G}) \leq \beta_{\text{DW}}(\mathcal{G})$.

For a given index coding problem $\mathcal{G}$, there exists at least one (weighted) alignment chain if and only if the problem is *half-rate-infeasible*, which means that the broadcast rate $\beta(\mathcal{G}) > 2$. For a problem $\mathcal{G}$ that is *half-rate-feasible*, we must have either that $\mathcal{G}$ is a complete graph with $\beta(\mathcal{G}) = 1$, or that $\beta(\mathcal{G}) = 2$, while for both cases the MAIS bound is known to be tight [2], [4], [11]. In the rest of the paper, unless otherwise specified, whenever we say an index coding problem $\mathcal{G}$, we assume that it is half-rate-infeasible.

## III. Motivating Result and Example

We identify a key drawback of the weighted alignment chain bounds of [9]. The proof is relegated to Appendix A.

*Proposition 2:* For a given index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, we have

$$\beta_{\text{MAIS}}(\mathcal{G}) \leq \beta_{\text{SW}}(\mathcal{G}) \leq \beta_{\text{DW}}(\mathcal{G}) \leq \beta_{\text{MAIS}}(\mathcal{G}) + 2/3. \tag{6}$$

Proposition 2 states that although the weighted alignment chain bounds are never looser than the MAIS bound for a given problem, they can at most improved the bound by the constant $\frac{2}{3}$, independent to the problem size $n$.

Moreover, the MAIS bound $\beta_{\text{MAIS}}(\mathcal{G})$ and broadcast rate $\beta(\mathcal{G})$ are both multiplicative under the lexicographic product of index coding side information graphs, and hence the gap between them is not fixed, but instead can be magnified to a multiplicative factor that grows polynomially in the problem size $n$ (See [2], [6]). A natural question is that whether we can find a way to improve the weighted alignment chain bounds such that the improved bound has similar properties.

Towards answering this question, we consider a motivating example in the following. First consider the following 5-message index coding problem $\mathcal{G}_0$,

$$(1|2,5), (2|1,3), (3|2,4), (4|3,5), (5|1,4), \tag{7}$$

whose side information graph is shown in Figure 1(a).

We have $\beta_{\text{MAIS}}(\mathcal{G}_0) = 2$, which is strictly looser than $\beta(\mathcal{G}_0) = \beta_{\text{SW}}(\mathcal{G}_0) = 2.5$, where $\beta_{\text{SW}}(\mathcal{G}_0)$ is obtained based on the following singleton weighted alignment chain [9]

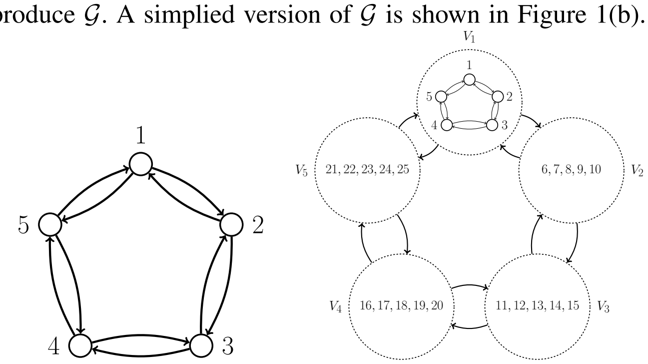
$$\underline{1} \overset{4}{\longleftrightarrow}_{\text{s}} 2 \overset{5}{\longleftrightarrow}_{\text{s}} \underline{3}, \tag{8}$$

as $\beta_{\text{SW}}(\mathcal{G}_0) = \frac{1+2m}{m} = \frac{5}{2}$, where $m = 2$ is the length of the chain.

Now consider the 25-message problem $\mathcal{G}$ whose side information graph is the lexicographic square of $\mathcal{G}_0$ discussed above, i.e., $\mathcal{G} = \mathcal{G}_0^{\circ 2} \doteq \mathcal{G} \circ \mathcal{G}$. For easier reference, set

$$V_i = \{5i-4, 5i-3, 5i-2, 5i-1, 5i,\}, \qquad \forall i \in [5].$$

and hence for $\mathcal{G}$, $[n] = \bigcup_{j\in[i]} V_i$. Note that for any $i \in [5]$, the subproblem $\mathcal{G}|_{V_i}$ can be seen as just a copy of the 5-message problem $\mathcal{G}_0$, used to replace the vertex $i$ in $\mathcal{G}_0$ to produce $\mathcal{G}$. A simplied version of $\mathcal{G}$ is shown in Figure 1(b).




(a) (b)

Figure 1. (a) The side information graph of $\mathcal{G}_0$ in (7). (b) A simplified version of the side information graph of $\mathcal{G} = \mathcal{G}_0^{\circ 2}$. We only draw $\mathcal{G}|_{V_1}$ in detail. An edge from a dashed circle $V_i$ to another dashed circle $V_j$ means that there is an edge from every node in $V_i$ to every node in $V_j$.

Since $\mathcal{G} = \mathcal{G}_0^{\circ 2}$, we have $\beta_{\text{MAIS}}(\mathcal{G}) = (\beta_{\text{MAIS}}(\mathcal{G}_0))^2 = 4$ and $\beta(\mathcal{G}) = (\beta(\mathcal{G}_0))^2 = 6.25$. According to Proposition 2, we have $\beta_{\text{SW}}(\mathcal{G}) \le \beta_{\text{DW}}(\mathcal{G}) \le \beta_{\text{MAIS}}(\mathcal{G}) + 2/3 = 4\frac{2}{3} < \beta(\mathcal{G}) = 6.25$. Therefore, we know that weighted alignment chains cannot give a tight result.

However, as the chain in (8) leads to the tight bound on $\mathcal{G}_0$, and $\mathcal{G} = \mathcal{G}_0^{\circ 2}$, if we allow a chain to be constituted by messages *sets* rather than just *individual* messages, we can construct the following chain

$$\underline{V_1} \overset{V_4}{\longleftrightarrow}_{\text{s}} V_2 \overset{V_5}{\longleftrightarrow}_{\text{s}} \underline{V_3}, \tag{9}$$

via replacing each component $i \in [5]$ in the chain in (8) by its corresponding $V_i$. Note that the chain in (9) satisfies that $V_1 \subseteq B_{V_3}, V_3 \subseteq B_{V_1}$, and that $V_1 \cup V_2 \subseteq B_{V_4}$, $V_2 \cup V_3 \subseteq B_{V_5}$.

The chain in (9) can also be presented in the following alternative form

$$\underline{(\underline{1} \overset{4}{\longleftrightarrow}_{\text{s}} 2 \overset{5}{\longleftrightarrow}_{\text{s}} \underline{3})} \xleftrightarrow{(\underline{16} \overset{19}{\longleftrightarrow}_{\text{s}} 17 \overset{20}{\longleftrightarrow}_{\text{s}} \underline{18})}_{\text{s}} (\underline{6} \overset{9}{\longleftrightarrow}_{\text{s}} 7 \overset{10}{\longleftrightarrow}_{\text{s}} \underline{8})$$
$$\xleftrightarrow{(\underline{21} \overset{24}{\longleftrightarrow}_{\text{s}} 22 \overset{25}{\longleftrightarrow}_{\text{s}} \underline{23})}_{\text{s}} \underline{(\underline{11} \overset{14}{\longleftrightarrow}_{\text{s}} 12 \overset{15}{\longleftrightarrow}_{\text{s}} \underline{13})},$$

which can be viewed as a *two-layer* chain, whose every component is a singleton weighted alignment chain. Using the similar techniques to the proof of the singleton weighted alignment chain in [9, Appendix 1] in a double layered fashion, it can be shown that the chain in (9) indeed leads to tight bound for the problem $\mathcal{G}$, and thus strictly outperforms any weighted alignment chain bound.

In next section, we formally propose a generalized version of the weighted alignment chain model, namely the *acyclic chain*, which is constituted by sets of messages.

## IV. Acyclic Chain Bounds

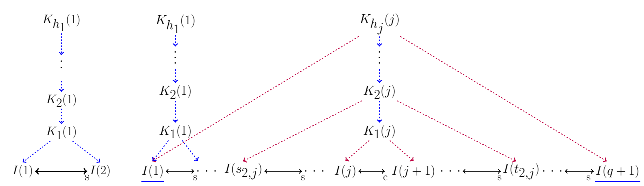
We start by defining the basic building block of the acyclic chains to be presented later, namely, the *basic acyclic tower*.

*Definition 2 (Basic Acyclic Tower):* For the index coding problem $\mathcal{G} : (i|A_i)$, $i \in [n]$, nonempty message sets $I(1), I(2), K_1(1), K_2(1), \cdots, K_{h_1}(1) \subseteq [n]$, constitute the following basic tower $\mathcal{B}_1$,

$$I(1) \overset{\substack{K_{h_1}(1)\\ \cdots \\ K_2(1) \\ K_1(1)}}{\longleftrightarrow}_{\text{s}} I(2),$$

if $I(1) \cup I(2) \cup K_1(1) \cup K_2(1) \cup \cdots K_{\ell-1}(1) \subseteq B_{K_\ell(1)}$ for any $\ell \in [h_1]$.

See Fig. 2(a) for a visualization of Definition 2. In the basic acyclic tower $\mathcal{B}_1$, message sets $I(1)$ and $I(2)$ are placed horizontally at the *ground level* of the tower, and message set $K_\ell(1)$ is placed on the $\ell$-th *floor* for any $\ell \in [h_1]$, where $h_1$ is called the *height* of the tower. The receivers in the message set on a higher floor must have all messages in the message sets located on lower floors, including the ground level, in their common interfering message set.




(a) (b)

Figure 2. Schematic graphs for (a) Definition 2 and (b) Definitions 3. To help with understanding, we draw dashed arrows such that if there is a directed path formed by dashed arrows of the same color from message sets $a$ to $b$, then $b \in B_a$. The purple arrows cannot criss-cross.

Based on the basic acyclic chain, we build the *crossing acyclic tower* as follows. For simpler notation, we use $\mathbf{K}_L(j)$ to denote the message sequence $K_\ell(j)$, $\ell \in L$ for some $j \in [m]$, $L \subseteq [h_j]$, e.g. $\mathbf{K}_{[h_j]}(j) = K_1(j), K_2(j), \cdots, K_{h_j}(j)$.

*Definition 3 (Crossing Acyclic Tower):* For the index coding problem $\mathcal{G} : (i|A_i)$, $i \in [n]$, we have the following crossing acyclic tower constituted by nonempty message sets,

$$I(1) \overset{\substack{K_{h_1}(1)\\ \cdots \\ K_2(1) \\ K_1(1)}}{\longleftrightarrow}_{\text{s}} \cdots I(j) \overset{\substack{K_{h_j}(j)\\ \cdots \\ K_2(j) \\ K_1(j)}}{\longleftrightarrow}_{\text{c}} \cdots \overset{\substack{K_{h_q}(q)\\ \cdots \\ K_2(q) \\ K_1(q)}}{\longleftrightarrow}_{\text{s}} I(q+1),$$

denoted as $\mathcal{X}_j$, if the conditions listed below are satisfied:

1) for any $j' \in [q] \setminus \{j\}$, the message sets $I(j'), I(j'+1), \mathbf{K}_{[h_{j'}]}(j')$ consitute a basic acyclic tower $\mathcal{B}_{j'}$;
2) for any $\ell \in [h_j]$, $K_1(j) \cup \cdots \cup K_{\ell-1}(j) \subseteq B_{K_\ell(j)}$;
3) for any $\ell \in [h_j]$ there exist two integers $s_{\ell,j} \in [j]$, $t_{\ell,j} \in [j+1 : q+1]$ such that
   a) $I(s_{\ell,j}) \cup I(t_{\ell,j}) \subseteq B_{K_\ell(j)}$;
   b) for any $\ell_1 < \ell_2 \in [h_j]$, we have $j = s_{1,j} \ge s_{\ell_1,j} \ge s_{\ell_2,j} \ge s_{h_j,j} = 1$, and $j+1 = t_{1,j} \le t_{\ell_1,j} \le t_{\ell_2,j} \le t_{h_j,j} = q+1$.

For the crossing acyclic tower $\mathcal{X}_j$ defined above, it has $q = q+1-1 = t_{h_j,j} - s_{h_j,j}$ edges. We call edge $j$ the *central* edge, and the collection of message sets $\{I(j), I(j+1), \mathbf{K}_{[h_j]}(j)\}$ the *core*. Every other edge $j' \in [q] \setminus \{j\}$ corresponds to a collection of message sets $\{I(j'), I(j'+1), \mathbf{K}_{[h_{j'}]}(j')\}$, which constitutes a basic acyclic tower $\mathcal{B}_{j'}$. Note that we use different subscripts for the edges in the horizontal chain in Definition 3 to distinguish the two different types of edges.

Also note that any basic acyclic tower $\mathcal{B}_{j'}$ itself can be seen as a special crossing acyclic tower with only the central edge and the core, for which $s_\ell(j') = j'$, $t_\ell(j') = j'+1$ for any $\ell \in [h_{j'}]$. In the rest of the paper, unless otherwise specified, whenever we say crossing acyclic tower we assume that it is not a basic acyclic tower.

Condition 3 in Definition 3 is described as follows. In the core, message set $K_\ell(j)$ on floor $\ell$ has message sets $I(s_{\ell,j})$ and $I(t_{\ell,j})$ to *start* and *terminate* its *coverage*, respectively. The coverage of a message set on a lower floor is within the range of the coverage of any other message set on a higher floor. For any tower with central edge $j$, the set of edges within the coverage of the message set on the top floor of the core is denoted as $G_j \doteq [s_{h_j}(j) : t_{h_j}(j) - 1]$. Then for any basic acyclic tower $\mathcal{B}_j$, $|G_j| = 1$, and for any crossing acyclic tower $\mathcal{X}_j$, $|G_j| \geq 2$.

For visualization of Definition 3, see Figure 2(b). To avoid clutter, we only draw $\mathcal{B}_1$ and the core of central edge $j$.

In the following we present the most general chain model in the paper, namely, the *disjoint acyclic chain*, which is built upon the basic and crossing acyclic towers.

*Definition 4 (Disjoint Acyclic Chain):* For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, we have the disjoint acyclic chain, $Ch$, of length $m$, shown as follows,

$$\underline{I(1)} \xleftrightarrow[K_1(1)]{K_{h_1}(1)\ \cdots} I(2) \xleftrightarrow[K_1(2)]{K_{h_2}(2)\ \cdots} \cdots \xleftrightarrow[K_1(m)]{K_{h_m}(m)\ \cdots} \underline{I(m+1)},$$

if the conditions listed below are satisfied:

1) $I(1) \subseteq B_{I(m+1)}$ or $I(m+1) \subseteq B_{I(1)}$;
2) For every $j \in [m]$, nonempty message sets $I(j), I(j+1), \mathbf{K}_{[h_j]}(j)$ constitute either a basic tower $\mathcal{B}_j$ or the core of a crossing tower $\mathcal{X}_j$;
3) Set $M \doteq \{j \in [m] : |G_j| \geq 2\}$ denote the set of central edges of the crossing towers within the chain, then for any $j_1 \neq j_2 \in M$, $G_{j_1} \cap G_{j_2} = \emptyset$.

We remove subscripts for the edges in the above definition as the positions of the basic and crossing towers are flexible.

Define $M' \doteq [m] \setminus (\bigcup_{j\in M} G_j)$ as the set of edges located outside the coverage of any crossing tower. Then the disjoint acyclic chain can be seen as a horizontal concatenation of the crossing towers $\mathcal{X}_j$, $j \in M$ and the basic towers $\mathcal{B}_{j'}$, $j' \in M'$, such that the terminals of the chain, $I(1), I(m+1)$, satisfy that $I(1) \in B_{I(m+1)}$ or $I(m+1) \in B_{I(1)}$. For a visualization of Definition 4, see Fig. 3.

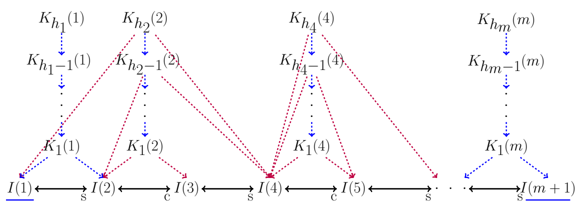


Figure 3. A visualization for Definition 4. If there is a directed path formed by dashed arrows of the same color from message sets $a$ to $b$, then $b \in B_a$. Definitions 3 and 4 jointly ensure that purple arrows can never criss-cross.

Given any disjoint acyclic chain $Ch$ as defined in Definition 4, every $I$-labeled and $K$-labeled sets in the chain is referred to as one of its *components*. Given a problem $\mathcal{G}$, let $\mathfrak{C}(\mathcal{G})$ denote the collection of its disjoint acyclic chains. Then $\mathfrak{C}(\mathcal{G}) = \emptyset$ if and only if $\mathcal{G}$ is half-rate-feasible.

For a given problem $\mathcal{G}$, we *recursively* define the lower bound $\Gamma(\mathcal{G})$, namely, *disjoint acyclic chain bound*.

*Definition 5 (Disjoint Acyclic Chain Bound):* Given an index coding problem $\mathcal{G} : (i|A_i)$, $i \in [n]$, define

$$\Gamma(\mathcal{G}) \doteq \max_{Ch \in \mathfrak{C}(\mathcal{G})} \Gamma(Ch),$$

where for any disjoint acyclic chain $Ch \in \mathfrak{C}(\mathcal{G})$,

$$\Gamma(Ch) \doteq \frac{1}{m}\Big(\sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(\mathcal{G}|_{K_\ell(j)}) + \sum_{j\in[m+1]} \Gamma(\mathcal{G}|_{I(j)})\Big),$$

with the following recursion termination conditions:

$$\Gamma(\mathcal{G}|_S) \doteq \beta_{\mathrm{MAIS}}(\mathcal{G}|_S), \quad \text{if } S \text{ is acyclic or } \mathfrak{C}(\mathcal{G}|_S) = \emptyset.$$

For brevity, when there is no ambiguity, for the subproblem $\mathcal{G}|_S$, we say subproblem $S$ or simply $S$. For example, $\Gamma(\mathcal{G}|_S)$ in the above definition can be simply denoted as $\Gamma(S)$.

*Remark 1:* The disjoint weighted alignment chain [9, Definition 5] can be seen as a special case of the disjoint acyclic chain defined in Definition 4, with the cardinality of every component fixed to be 1. For any disjoint weighted alignment chain $Ch$, $\Gamma(Ch) = \frac{1}{m}(\sum_{j\in[m]}\sum_{\ell\in[h_j]} 1 + \sum_{j\in[m+1]} 1) = \frac{\sum_{j\in[m]} h_j + m + 1}{m} = \beta_{\mathrm{DW}}(Ch)$.

The main result of this section is stated as follows.

*Theorem 1:* For the index coding problem $\mathcal{G}$: $(i|A_i)$, $i \in [n]$, its broadcast rate $\beta(\mathcal{G})$ is lower bounded by $\Gamma(\mathcal{G})$, i.e.,

$$\Gamma(\mathcal{G}) \leq \beta(\mathcal{G}). \qquad (10)$$

To prove Theorem 1, we define a *s-layer* chain or problem.

*Definition 6:* For any integer $s \geq 0$, a $s$-layer disjoint acyclic chain $Ch$ or problem $\mathcal{G}$ is defined as follows.

1) We say a problem $\mathcal{G}$ is 0-layer if $\mathfrak{C}(\mathcal{G}) = \emptyset$, i.e., $\mathcal{G}$ is not half-rate-infeasible.
2) We say a chain $Ch$ is 1-layer, if its components are all acyclic or not half-rate-infeasible. We say a problem $\mathcal{G}$ is 1-layer if it has at least one valid chain and all its valid chains $Ch \in \mathfrak{C}(\mathcal{G})$ are 1-layer.
3) We say a chain $Ch$ is 2-layer, if its components are all at most 1-layer, and there is at least one component that is half-rate-feasible or not acyclic. We say a problem $\mathcal{G}$ is 2-layer if it has at least one 2-layer chain and all its valid chains $Ch \in \mathfrak{C}(\mathcal{G})$ are at most 2-layer.
4) We say a chain $Ch$ is 3-layer, if its components are all at most 2-layer, and there is at least one 2-layer component. We say a problem $\mathcal{G}$ is 3-layer if it has at least one 3-layer chain and all its valid chains $Ch \in \mathfrak{C}(\mathcal{G})$ are at most 3-layer.
5) A chain or a problem that is of $s$-layer, for any integer $s > 3$, can be defined so on and so forth.

For example, the chain in (9) is a 2-layer acyclic chain as its each component $V_i$, $i \in [5]$ is an 1-layer subproblem.

The lemma below is the key to showing Theorem 1, which can be proved using mathematical induction on the number of layers of subproblem $Q$ as defined in Definition 6.

*Lemma 1:* For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, let $R$ be any achievable symmetric rate. For any nonempty set $Q \subseteq [n]$, we have

$$g(Q,P) \geq g(P) + R \cdot \Gamma(Q), \qquad \forall P \subseteq B_Q. \tag{11}$$

The proof of Lemma 1 is presented in Appendix B. Now we prove Theorem 1 using Lemma 1.

*Proof:* Proving that $\Gamma(\mathcal{G}) \leq \beta(\mathcal{G})$ is equivalent to proving that for any achievable symmetric rate $R$, we have $R \cdot \Gamma(\mathcal{G}) \leq 1$, which is shown in the following.

Set $Q = [n]$, $P = \emptyset = B_Q$. Since that $g(S) \leq 1, \forall S \subseteq [n]$, and that $g(P) = g(\emptyset) = 0$ and $\mathcal{G}|_Q = \mathcal{G}|_{[n]} = \mathcal{G}$, we have

$$1 \geq g(Q,P) \geq g(P) + R \cdot \Gamma(Q) = R \cdot \Gamma(Q) = R \cdot \Gamma(\mathcal{G}),$$

where the second inequality follows from Lemma 1. ■

*Example 1:* Recall the 25-message problem $\mathcal{G}$ in Section III with the 2-layer disjoint acyclic chain $Ch$ as shown in (9). By Definition 5 and Theorem 1, we have

$$\Gamma(Ch) = \frac{1}{2}(\sum_{i \in [5]} \Gamma(V_i)) = 6.25 \leq \Gamma(\mathcal{G}) \leq \beta(\mathcal{G}) = 6.25,$$

which means that the disjoint acyclic bound $\Gamma(\mathcal{G})$ is indeed tight for the problem, and thus strictly outperforms the disjoint weighted alignment chain bound $\beta_{\mathrm{DW}}(\mathcal{G})$.

Recall that any disjoint acyclic chain can be seen as a concatenation of some crossing acyclic towers and basic acyclic towers. If a disjoint acyclic chain is constructed by basic acyclic towers only, it reduces to a *singleton acyclic chain*, defined as follows.

*Definition 7 (Singleton Acyclic Chain):* For the index coding problem $\mathcal{G} : (i|A_i)$, $i \in [n]$, we have the singleton acyclic chain, $Ch_{\mathrm{s}}$, of length $m$, shown as follows,

$$\underline{I(1)} \xleftrightarrow[\ ]{\substack{K_{h_1}(1)\\ \cdots \\ K_1(1)}}_{\mathrm{s}} I(2) \xleftrightarrow[\ ]{\substack{K_{h_2}(2)\\ \cdots \\ K_1(2)}}_{\mathrm{s}} \cdots \xleftrightarrow[\ ]{\substack{K_{h_m}(m)\\ \cdots \\ K_1(m)}}_{\mathrm{s}} \underline{I(m+1)},$$

if the conditions listed below are satisfied:

1) $I(1) \subseteq B_{I(m+1)}$ or $I(m+1) \subseteq B_{I(1)}$;
2) for any $j \in [m]$, nonempty message sets $I(j), I(j+1), \mathbf{K}_{[h_j]}(j)$ form a basic acyclic tower $\mathcal{B}_j$.

Definition 5 and Theorem 1 can be readily modified to the singleton acyclic chain version. In particular, for the problem $\mathcal{G}$, we use $\Gamma_{\mathrm{s}}(\mathcal{G})$ to denote the lower bound on the broadcast rate given by the singleton acyclic chains.

We have the following theorem.

*Theorem 2:* For the index coding problem $\mathcal{G}$, we have

$$\beta_{\mathrm{SW}}(\mathcal{G}) \leq \Gamma_{\mathrm{s}}(\mathcal{G}) \leq \Gamma(\mathcal{G}), \qquad \beta_{\mathrm{DW}}(\mathcal{G}) \leq \Gamma(\mathcal{G}).$$

Note that $\beta_{\mathrm{DW}}(\mathcal{G})$ and $\Gamma_{\mathrm{s}}(\mathcal{G})$ can outperform each other.

## V. Properties of The Acyclic Chain Bounds

We identify important properties of the disjoint acyclic chain bound $\Gamma(\mathcal{G})$. Same properties hold for the less general singleton acyclic bound. We start with the following theorem.

*Theorem 3:* For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, for any set $P, Q \subseteq [n]$ such that $P \subseteq B_Q$, we have

$$\Gamma(P \cup Q) \geq \Gamma(P) + \Gamma(Q). \tag{12}$$

The proof of Theorem 3 is provided in Appendix C.

Similar to the PM lower bound $\beta_{\mathrm{PM}}(\mathcal{G})$ (See, for example, [2, Proposition 5.4]), the disjoint acyclic chain lower bound $\Gamma(\mathcal{G})$ has the following structural property under the lexicographic product of side information graphs.

*Theorem 4:* For two given problems $\mathcal{G}_0$ and $\mathcal{G}_1$, we have

$$\Gamma(\mathcal{G}_0 \circ \mathcal{G}_1) \geq \Gamma(\mathcal{G}_0) \cdot \Gamma(\mathcal{G}_1). \tag{13}$$

We use mathematical induction on the number of layers of $\mathcal{G}_0$ to prove Theorem 4, which is presented in Appendix D.

The corollary below states that the gap between the disjoint acyclic chain bound $\Gamma(\mathcal{G})$ and the MAIS bound $\beta_{\mathrm{MAIS}}(\mathcal{G})$ can be magnified to a multiplicative factor which grows polynomially in the problem size $n$.

*Corollary 1:* Let $\mathcal{G}_0$ be an index coding problem with $n_0 = V(\mathcal{G}_0)$ messages for which $\Gamma(\mathcal{G}_0)/\beta_{\mathrm{MAIS}}(\mathcal{G}_0) = \rho > 1$. Consider index coding problem $\mathcal{G} = \mathcal{G}^{\circ k}$ with $n = V(\mathcal{G}) = n_0^k$ messages for any positive integer $k$, we have

$$\frac{\Gamma(\mathcal{G})}{\beta_{\mathrm{MAIS}}(\mathcal{G})} \geq n^{\log_{n_0}(\rho)}.$$

*Proof:* By Theorem 4 and [2, Proposition 5.1], we have $\frac{\Gamma(\mathcal{G})}{\beta_{\mathrm{MAIS}}(\mathcal{G})} \geq \frac{(\Gamma(\mathcal{G}_0))^k}{(\beta_{\mathrm{MAIS}}(\mathcal{G}))^k} = (n_0^{\log_{n_0}(\rho)})^k = n^{\log_{n_0}(\rho)}$. ■

## Appendix A
## Proof of Proposition 2

*Proof:* We only show that $\beta_{\mathrm{DW}}(\mathcal{G}) \leq \beta_{\mathrm{MAIS}}(\mathcal{G}) + 2/3$ as the rest of (6) has been shown in [9]. We do not repeat the definition of the disjoint weighted alignment chain and its corresponding bound (see [9, Definition 5, Theorem 2]). Assume that $\beta_{\mathrm{DW}}(\mathcal{G})$ is given by some disjoint weighted alignment chain shown as follows,

$$\underline{i(1)} \xleftrightarrow{\substack{k_{h_1}(1)\\ \cdots \\ k_1(1)}} i(2) \xleftrightarrow{\substack{k_{h_2}(2)\\ \cdots \\ k_1(2)}} \cdots \xleftrightarrow{\substack{k_{h_m}(m)\\ \cdots \\ k_1(m)}} \underline{i(m+1)}.$$

When $m = 1$, by [9, Definition 5], the message set $\{i(1), i(2), k_1(1), \cdots, k_{h_1}(1)\}$ is acyclic, and thus we have

$$\beta_{\mathrm{DW}}(\mathcal{G}) = |\{i(1), i(2), k_1(1), \cdots, k_{h_1}(1)\}| \leq \beta_{\mathrm{MAIS}}(\mathcal{G}).$$

When $m \geq 2$, by [9, Definition 5], for any edge $j \in [m]$, the message set $\{i(j), i(j+1), k_1(j), \cdots, k_{h_j}(j)\}$ constitutes either a basic tower or the core of a crossing tower. The message set $\{k_1(j), \cdots, k_{h_j}(j)\}$ is always acyclic, which indicates that $h_j = |\{k_1(j), \cdots, k_{h_j}(j)\}| \leq \beta_{\mathrm{MAIS}}(\mathcal{G})$.

If $\{i(j), i(j+1), k_1(j), \cdots, k_{h_j}(j)\}$ constitutes a basic tower, then the message set $\{i(j), k_1(j), \cdots, k_{h_j}(j)\}$ is acyclic, and thus $h_j \leq \beta_{\mathrm{MAIS}}(\mathcal{G}) - 1$.

If $\{i(j), i(j+1), k_1(j), \cdots, k_{h_j}(j)\}$ is the core of a crossing tower, i.e., $j \in M = \{j \in [m] : |G_j| \geq 2\}$, we show that $h_j = \beta_{\mathrm{MAIS}}(\mathcal{G})$ only if $|G_j| \geq 3$. Given $h_j = \beta_{\mathrm{MAIS}}(\mathcal{G})$, we must have $s_{h_j,j} \leq j-1$ since otherwise $j = s_{\ell,j} \in B_{k_\ell(j)}$ for any $\ell \in [h_j]$, and subsequently $\{i(j), k_1(j), \cdots, k_{h_j}(j)\}$ is acyclic and $h_j \leq \beta_{\mathrm{MAIS}}(\mathcal{G}) - 1$. Similarly, given $h_j = \beta_{\mathrm{MAIS}}(\mathcal{G})$, we must have $t_{h_j,j} \geq j+2$, which, together with that $s_{h_j,j} \leq j-1$, yield that $|G_j| \geq 3$. Also note that for any $j_1 \neq j_2 \in M$, $G_{j_1} \cap G_{j_2} = \emptyset$. Therefore, there is at most $\lfloor \frac{m}{3} \rfloor$ many $j$ such that $h_j = \beta_{\mathrm{MAIS}}(\mathcal{G})$, while for all other $j \in [m]$, $h_j \leq \beta_{\mathrm{MAIS}}(\mathcal{G}) - 1$. As a result, we have

$$\beta_{\mathrm{DW}}(\mathcal{G}) = \frac{1}{m}(1 + m + h_1 + h_2 + \cdots + h_m)$$
$$\leq \frac{1}{m}(1 + m\beta_{\mathrm{MAIS}}(\mathcal{G}) + \lfloor \frac{m}{3} \rfloor) \leq \beta_{\mathrm{MAIS}}(\mathcal{G}) + \frac{2}{3},$$

where the last inequality is due to that when $m = 2$, $\frac{1}{m} + \frac{\lfloor \frac{m}{3} \rfloor}{m} = \frac{1}{2} < \frac{2}{3}$ and otherwise $\frac{1}{m} + \frac{\lfloor \frac{m}{3} \rfloor}{m} \leq \frac{1}{3} + \frac{m/3}{m} = \frac{2}{3}$. ■

## APPENDIX B
## PROOF OF LEMMA 1

Consider the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$. Define set function

$$g(S) \doteq \frac{1}{r} H(Y_{[n]}|\mathbf{X}_{S^c}), \qquad \forall S \subseteq [n].$$

Clearly, $g(\emptyset) = 0$. It can be verified that for any $S, S' \subseteq [n]$, the set function $g(S)$ has the following two properties,

$$g(S) \le g(S'), \qquad \text{if } S \subseteq S',$$
$$g(S \cap S') + g(S \cup S') \le g(S) + g(S'),$$

which we refer to as the monotonicity and submodularity of $g(S)$, respectively. Also, $g(S) \le 1, \forall S \subseteq [n]$.

For simplicity, we use $j$ to denote $\{j\}$ when there is no ambiguity. Also, for any sets $S_1, S_2, \cdots \subseteq [n]$, we use $g(S_1, S_2, \cdots)$ to denote $g(S_1 \cup S_2 \cup \cdots)$. For example, for $n = 3$, $g(\{1, 2\} \cup \{3\})$ and $g(\{1, 2\}, 3)$ mean the same thing.

We have the following lemma from [8], referred to as the *single-receiver decoding lemma*.

*Lemma 2 ( [8]):* For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, for any $i \in [n]$, we have

$$R_i + g(B) = g(B \cup \{i\}), \qquad \forall B \subseteq B_i. \tag{14}$$

Particularly, when $B = \emptyset$, $g(B) = 0$, and thus $R_i = g(\{i\})$.

We show a more general *multi-receiver decoding lemma*.

*Lemma 3:* For the index coding problem $\mathcal{G} : (i|A_i), i \in [n]$, consider any two sets $K, K' \subseteq [n]$ such that $K' \subseteq B_K$. If $K$ is acyclic or subproblem $K$ is half-rate-feasible, then

$$g(K, K') = R \cdot \Gamma(K) + g(K'). \tag{15}$$

*Proof:* The proof for the case when $\mathcal{G}|_K$ is half-rate-infeasible is relatively straightforward and thus omitted due to limited space. Consider the case when $K$ is acyclic. Given that $K$ is acyclic and that $K' \subseteq B_K$, there exists an ordering of elements in $K$ as $k_1, k_2, \cdots, k_{|K|}$ such that $\{k_j : j \in [\ell - 1]\} \cup K' \subseteq B_{k_\ell}$ for any $\ell \in [|K|]$. Hence, by Lemma 2,

$$\begin{aligned} g(K, K') &= g(k_1, k_2, \cdots, k_{|K|-2}, k_{|K|-1}, k_{|K|}, K') \\ &= R_{k_{|K|}} + g(k_1, k_2, \cdots, k_{|K|-2}, k_{|K|-1}, K') \\ &= R_{k_{|K|}} + R_{k_{|K|-1}} + g(k_1, k_2, \cdots, k_{|K|-2}, K') \\ &= \cdots \\ &= \sum_{i \in K} R_i + g(K') = R \cdot \Gamma(K) + g(K'), \end{aligned}$$

where the last equality is due to that $\Gamma(K) = |K|$ for any acyclic $K$. ■

Now we prove Lemma 1 via mathematical induction.

Due to limited space, we only provide detailed proof for the induction base that (11) holds where $Q$ is at most 1-layer, utilizing the properties of $g(S)$ and the multi-receiver decoding lemma, Lemma 3. The inductive step can be shown following similar steps to the proof of the induction base, utilizing the properties of $g(S)$, as well as the induction hypothesis instead of Lemma 3.

*Proof:* We prove the induction base that (11) holds when $Q$ is at most 1-layer, which can be divided into two subcases:

1) When $Q$ is 0-layer, i.e.,$Q$ is half-rate-feasible, (11) directly follows from Lemma 3.

2) When $Q$ is 1-layer, assume that the disjoint acyclic chain $Ch \in \mathfrak{C}(Q)$, as defined in Definition 4, satisfies that $\Gamma(Q) = \Gamma(Ch)$. Since that $Q$ is an 1-layer problem, $Ch$ must be 1-layer and its components must be acyclic or half-rate-feasible.

Consider any crossing tower within $Ch$, $\mathcal{X}_j$, $j \in M$.

Within $\mathcal{X}_j$, consider any basic tower $\mathcal{B}_{j'}$, $j' \in G_j \setminus \{j\}$ constitued by sets $I(j'), I(j'+1), \mathbf{K}_{[h_{j'}]}(j')$. By Definition 2 and that $P \subseteq B_Q \subseteq B_{K_\ell(j')}, \forall \ell \in [h_{j'}]$, we have

$$\begin{aligned} &I(j') \cup I(j'+1) \cup P \cup K_1(j') \cup \cdots \cup K_{\ell-1}(j') \\ &\quad \subseteq B_{K_\ell(j')}, \qquad \forall \ell \in [h_{j'}]. \end{aligned} \tag{16}$$

Hence, we have

$$\begin{aligned} g(Q, P) &\ge g(\mathbf{K}_{[h_{j'}]}(j'), I(j'), I(j'+1), P) \\ &= R \cdot \sum_{\ell \in [h_{j'}]} \Gamma(K_\ell(j')) + g(I(j'), I(j'+1), P), \end{aligned} \tag{17}$$

where the inequality follows from the monotonicty of $g(S)$ and the equality follows from repeated application of Lemma 3 given (16) and that every component of $Ch$ is acyclic or half-rate-feasible. Adding up (17) for all $j' \in G_j \setminus \{j\}$ yields

$$\begin{aligned} &\sum_{j' \in G_j \setminus \{j\}} (R \cdot \sum_{\ell \in [h_{j'}]} \Gamma(K_\ell(j')) + g(I(j'), I(j'+1), P)) \\ &\le \sum_{j' \in G_j \setminus \{j\}} g(Q, P) = (|G_j| - 1) \cdot g(Q, P). \end{aligned} \tag{18}$$

Now consider the core of $\mathcal{X}_j$. When there is no ambiguity, for any $\ell \in [h_j]$, we use $s_\ell$ and $t_\ell$ to denote $s_{\ell,j}$ and $t_{\ell,j}$, respectively. For any $\ell \in [h_j]$, by Definition 3 and that $P \subseteq B_Q \subseteq B_{K_\ell(j)}$, we have

$$I(s_\ell) \cup I(t_\ell) \cup P \cup K_1(j) \cdots \cup K_{\ell-1}(j) \subseteq B_{K_\ell(j)}. \tag{19}$$

Hence, for any $\ell \in [2 : h_j]$, we have

$$\begin{aligned} &g(\mathbf{K}_{[\ell-2]}(j), I(s_{\ell-1}), I(t_{\ell-1}), P) + R \cdot \Gamma(K_{\ell-1}(j)) \\ &= g(\mathbf{K}_{[\ell-1]}(j), I(s_{\ell-1}), I(t_{\ell-1}), P) \end{aligned} \tag{20}$$

$$\begin{aligned} &\le g(\mathbf{K}_{[\ell-1]}(j), I(s_\ell), I(t_\ell), P) - g(I(s_\ell), I(t_\ell), P) \\ &\quad + g(I(s_{\ell-1}), I(t_{\ell-1}), I(s_\ell), I(t_\ell), P), \end{aligned} \tag{21}$$

where (20) follows from Lemma 3 given (19) and that every component of $Ch$ is acyclic or half-rate-feasible, and the inequality follow from the monotonicity and submodularity of $g(S)$. By reasoning similar to that of (20) and the monotonicity of $g(S)$, we have

$$\begin{aligned} &g(\mathbf{K}_{[h_j-1]}(j), I(s_{h_j}), I(t_{h_j}), P) + R \cdot \Gamma(K_{h_j}(j)) \\ &= g(\mathbf{K}_{[h_j]}(j), I(s_{h_j}), I(t_{h_j}), P) \le g(Q, P). \end{aligned} \tag{22}$$

Summing up (21) for all $\ell \in [2 : h_j]$ and adding (22), and then eliminating redundant terms gives

$$\begin{aligned} &\sum_{\ell \in [h_j]} g(I(s_\ell), I(t_\ell), P) + R \cdot \sum_{\ell \in [h_j]} \Gamma(K_\ell(j)) - g(Q, P) \\ &\le \sum_{\ell \in [2:h_j]} g(I(s_{\ell-1}), I(t_{\ell-1}), I(s_\ell), I(t_\ell), P). \end{aligned} \tag{23}$$

Next consider any $a \le b \in G_j$ within the coverage of $\mathcal{X}_j$. For any $c \in [a + 1 : b]$, we have

$$\begin{aligned} &g(I(a), I(c), P) + g(I(c), I(c+1), P) \\ &\ge g(P) + R \cdot \Gamma(I(c)) + g(I(a), I(c+1), P), \end{aligned} \tag{24}$$

where the inequality follows from the submodularity and monotonicity of $g(S)$, as well as Lemma 3 with $B \subseteq B_Q \subseteq I(c)$ and that every component of $Ch$ is acyclic or half-rate-feasible. Summing up (24) for all $c \in [a+1:b]$ yields

$$\sum_{c\in[a:b]} g(I(c), I(c+1), P) \geq$$
$$\sum_{c=[a+1:b]} (g(P) + R\cdot\Gamma(I(c))) + g(I(a), I(b+1), P). \quad (25)$$

Consider any $\ell \in [2:h_j]$. Using (25) and the submodularity of $g(S)$, as well as Lemma 3, we have

$$\sum_{\substack{j'\in[s_\ell:s_{\ell-1}-1]\\ \cup[t_{\ell-1}:t_\ell-1]}} g(I(j'), I(j'+1), P) + g(I(s_{\ell-1}), I(t_{\ell-1}), P)$$
$$\geq \sum_{j'\in[s_\ell+1:s_{\ell-1}]\cup[t_{\ell-1}:t_\ell-1]} (R\cdot\Gamma(I(j')) + g(P))$$
$$+ g(I(s_\ell), I(s_{\ell-1}), I(t_{\ell-1}), I(t_\ell), P). \quad (26)$$

Definition 3 states that for any $\ell \in [2:h_j]$, $s_\ell \leq s_{\ell-1} \leq s_1 = j$, and $j+1 = t_1 = t_{\ell-1} \leq t_\ell$. Hence, adding up (26) for all $\ell \in [2:h_j]$ and rearranging yields

$$\sum_{\ell\in[2:h_j]} g(I(s_\ell), I(s_{\ell-1}), I(t_{\ell-1}), I(t_\ell), P)$$
$$+ \sum_{j'\in[s_{h_j}+1:t_{h_j}-1]} (R\cdot\Gamma(I(j')) + g(P))$$
$$- \sum_{j'\in G_j\setminus\{j\}} g(I(j'), I(j'+1), P)$$
$$\leq \sum_{\ell\in[2:h_j]} g(I(s_{\ell-1}), I(t_{\ell-1}), P). \quad (27)$$

Summing up (18), (23), and (27) and simplifying the result yields that for the crossing tower $\mathcal{X}_j$, we have

$$R\cdot\sum_{j'\in G_j}\sum_{\ell\in[h_{j'}]} \Gamma(K_\ell(j')) + g(I(s_{h_j}), I(t_{h_j}), P)$$
$$+ \sum_{j'\in[s_{h_j}+1:t_{h_j}-1]} (R\cdot\Gamma(I(j')) + g(P))$$
$$\leq |G_j|\cdot g(Q, P). \quad (28)$$

Recall that $Ch$ can be seen as a concatenation of the crossing towers $\mathcal{X}_j$, $j \in M$, and the basic towers $\mathcal{B}_{j'}$, $j' \in M' = [m]\setminus(\bigcup_{j\in M} G_j)$. By (28) and (17), we have

$$\sum_{j\in M}(R\cdot\sum_{j'\in G_j}\sum_{\ell\in[h_{j'}]} \Gamma(K_\ell(j')) + g(I(s_{h_j,j}), I(t_{h_j,j}), P)$$
$$+ \sum_{j'\in[s_{h_j,j}+1:t_{h_j,j}-1]} (R\cdot\Gamma(I(j')) + g(P)))$$
$$+ \sum_{j'\in M'} (R\cdot\sum_{\ell\in[h_{j'}]} \Gamma(K_\ell(j')) + g(I(j'), I(j'+1), P))$$
$$\leq \sum_{j\in M}(|G_j|\cdot g(Q,P)) + \sum_{j'\in M'} g(Q,P). \quad (29)$$

For the RHS of (29), we have

$$RHS = g(Q,P)(\sum_{j\in M}|G_j| + |[m]\setminus(\bigcup_{j\in M} G_j)|)$$
$$= m\cdot g(Q,P), \quad (30)$$

where the last equality is due to the fact that for any $j_1 \neq j_2 \in [m]$, $G_{j_1}\cap G_{j_2} = \emptyset$ by Definition 4.

Since that any basic tower $\mathcal{B}_{j'}$, $j' \in M'$ can be seen as a special crossing tower with $s_{h_{j'},j'} = j'$, $t_{h_{j'},j'} = j'+1$, we can rearrange the LHS of (29) as follows,

$$LHS = R\cdot\sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(K_\ell(j))$$
$$+ \sum_{j\in M}\sum_{j'\in[s_{h_j,j}+1:t_{h_j,j}-1]} (R\cdot\Gamma(I(j')) + g(P))$$
$$+ \sum_{j\in M\cup M'} g(I(s_{h_j,j}), I(t_{h_j,j}), P), \quad (31)$$

Set $M\cup M'$ denotes the collection of central edges of the crossing towers $\mathcal{X}_j$, $j\in M$ and the basic towers $\mathcal{B}_{j'}$, $j'\in M'$. As these towers are concatenated, if we order the elements of $M\cup M'$ as $j_1 < j_2 < \cdots < j_{|M\cup M'|}$, we have

$$s_{h_{j_1},j_1} = 1, \quad (32)$$
$$t_{h_{j_{|M\cup M'|}},j_{|M\cup M'|}} = m+1, \quad (33)$$
$$s_{h_{j_v},j_v} = t_{h_{j_{v-1}},j_{v-1}}, \qquad \forall v\in[2:|M\cup M'|]. \quad (34)$$

By the submodularity and the monotonicity of $g(S)$, and Lemma 3 with $P\subseteq B_Q\subseteq B_{I(s_{h_{j_v}},j_v)}$ and $I(s_{h_{j_v},j_v})$ being acyclic or half-rate-feasible, for any $v\in[2:|M\cup M'|]$,

$$g(I(s_{h_{j_1},j_1}), I(s_{h_{j_v},j_v}), P) + g(I(s_{h_{j_v},j_v}), I(t_{h_{j_v},j_v}), P)$$
$$\geq g(I(s_{h_{j_1},j_1}), I(t_{h_{j_v},j_v}), P) + R\cdot\Gamma(I(s_{h_{j_v},j_v})) + g(P).$$

Given (32)-(34), as well as the fact that $j_v$, $v\in[|M\cup M'|]$ is a reindexing of $j$, $j\in M\cup M'$, summing up the above inequality for all $v\in[2:|M\cup M'|]$ and simplifying yields

$$\sum_{j\in M\cup M'} g(I(s_{h_j,j}), I(t_{h_j,j}), P)$$
$$\geq \sum_{v\in[2:|M\cup M'|]} (R\cdot\Gamma(I(s_{h_{j_v},j_v})) + g(P))$$
$$+ g(I(s_{h_{j_1},j_1}), I(t_{h_{j_{|M\cup M'|}},j_{|M\cup M'|}}), P)$$
$$= \sum_{j'\in(\bigcup_{j\in M\cup M'}\{s_{h_j,j},t_{h_j,j}\})\setminus\{1,m+1\}} (R\cdot\Gamma(I(j')) + g(P))$$
$$+ g(I(1), I(m+1), P)$$
$$= \sum_{j'\in\bigcup_{j\in M\cup M'}\{s_{h_j,j},t_{h_j,j}\}} (R\cdot\Gamma(I(j')) + g(P)) - g(P), \quad (35)$$

where the last equality is due to that

$$g(I(1), I(m+1), P) = g(P) + R\cdot\sum_{j'\in\{1,m+1\}} \Gamma(I(j'))$$

which can be shown by applying Lemma 3 twice given that $P\subseteq B_Q\subseteq B_{I(j)}$ for any $j'\in\{1,m+1\}$, that $I(1)\subseteq B_{I(m+1)}$ or $I(m+1)\subseteq B_{I(1)}$ by Definition 4, and that every component of $Ch$ is acyclic or half-rate-feasible.

Combinning (31) and (35), we bound the LHS of (29) as

$$LHS \geq R\cdot\sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(K_\ell(j))$$
$$+ \sum_{j\in M}\sum_{j'\in[s_{h_j,j}+1:t_{h_j,j}-1]} (R\cdot\Gamma(I(j')) + g(P))$$
$$+ \sum_{j'\in\bigcup_{j\in M\cup M'}\{s_{h_j,j},t_{h_j,j}\}} (R\cdot\Gamma(I(j')) + g(P)) - g(P)$$
$$= R\cdot(\sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(K_\ell(j)) + \sum_{j\in[m+1]} \Gamma(I(j))) + m\cdot g(P)$$
$$= m\cdot R\cdot\Gamma(Ch) + m\cdot g(P)$$

$$=m \cdot R \cdot \Gamma(Q) + m \cdot g(P), \quad (36)$$

where the second last equality follows from Definition 5, and the last equality follows from that $\Gamma(Q) = \Gamma(Ch)$.

Finally, given (29), (30) and (36), we can conclude that $g(Q, P) \geq g(P) + R \cdot \Gamma(Q)$. ■

## Appendix C
## Proof of Theorem 3

*Proof:* Assume that for the subproblem $P$, the disjoint acyclic chain $Ch \in \mathfrak{C}(P)$ as shown in Definition 4 satisfies that $\Gamma(P) = \Gamma(Ch)$. Since $P \subseteq B_Q$, we can construct a chain $Ch^Q$ based on $Ch$ as:

$$\underline{I(1)} \overset{\overset{K^Q_{h_1}(1)}{\cdots}}{\underset{}{\xleftrightarrow{K_1(1)}}}_{\mathrm{s}} I(2) \overset{\overset{K^Q_{h_2}(2)}{\cdots}}{\xleftrightarrow{K_1(2)}}_{\mathrm{s}} \cdots \overset{\overset{K^Q_{h_m}(m)}{\cdots}}{\xleftrightarrow{K_1(m)}}_{\mathrm{s}} \underline{I(m+1)},$$

such that $Ch^Q \in \mathfrak{C}(P \cup Q)$. We have

$$\begin{aligned}
&\Gamma(Ch^Q) \\
=&\frac{1}{m}(\sum_{j\in[m]} \Gamma(Q) + \sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(K_\ell(j)) + \sum_{j\in[m+1]} \Gamma(I(j))) \\
=&\Gamma(Q) + \Gamma(P) = \Gamma(Q) + \Gamma(Ch), \quad (37)
\end{aligned}$$

where the first and second equalities follow from Definition 5. Since $Ch^Q \in \mathfrak{C}(P \cup Q)$, by Definition 5, we also have

$$\Gamma(P \cup Q) = \max_{Ch' \in \mathfrak{C}(P\cup Q)} \Gamma(Ch') \geq \Gamma(Ch^Q). \quad (38)$$

Combinning (37) and (38) leads to (12). ■

## Appendix D
## Proof of Theorem 4

*Lemma 4:* For any $\mathcal{G}_0$, $\mathcal{G}_1$, consider any $P, Q \subseteq V(\mathcal{G}_0)$ such that $P \subseteq B_Q$. Then for $\mathcal{G} = \mathcal{G}_0 \circ \mathcal{G}_1$ where $V(P \circ \mathcal{G}_1), V(Q \circ \mathcal{G}_1) \subseteq V(\mathcal{G})$, we have $V(P \circ \mathcal{G}_1) \subseteq B_{V(Q\circ\mathcal{G}_1)}$.

*Proof:* By the definition of the common interfering message set and Definition 1, for any $P, Q \subseteq V(\mathcal{G}_0)$, $P \subseteq B_Q$, we know that there is no edge going from any node in $P$ to any node in $Q$ in the directed graph $\mathcal{G}_0$, and hence there is no edge going from any node in $V(P \circ \mathcal{G}_1)$ to any node in $V(Q \circ \mathcal{G}_1)$, which indicates that $V(P \circ \mathcal{G}_1) \subseteq B_{V(Q\circ\mathcal{G}_1)}$. ■

*Lemma 5:* For any $\mathcal{G}_0$, $\mathcal{G}_1$, if $\mathcal{G}_0$ is acyclic or half-rate-feasible, then

$$\Gamma(\mathcal{G}_0 \circ \mathcal{G}_1) \geq \Gamma(\mathcal{G}_0) \cdot \Gamma(\mathcal{G}_1). \quad (39)$$

*Proof:* The proof for (39) when $\mathcal{G}_0$ is half-rate-feasible is relatively straightforward and thus omitted. Consider the case when $\mathcal{G}_0$ is acyclic. Let $n_0 = |V(\mathcal{G}_0)|$, and there exists an ordering of elements in $V(\mathcal{G}_0)$, denoted as $v_1, v_2, \cdots, v_{n_0}$ such that $v_1 \cup v_2 \cup \cdots \cup v_{i-1} \subseteq B_{v_i}$ for any $i \in [n_0]$. Hence, by Lemma 4, we have

$$V(\{v_j : j \in [i-1]\} \circ \mathcal{G}_1) \subseteq B_{V(v_i\circ\mathcal{G}_1)}, \quad \forall i \in [n_0]. \quad (40)$$

Given (40), repeatedly applying Theorem 3 yields

$$\begin{aligned}
&\Gamma(\mathcal{G}_0 \circ \mathcal{G}_1) \\
&\geq \Gamma(\{v_1, \cdots, v_{n_0-1}\} \circ \mathcal{G}_1) + \Gamma(v_{n_0} \circ \mathcal{G}_1) \\
&\geq \Gamma(\{v_1, \cdots, v_{n_0-2}\} \circ \mathcal{G}_1) + \Gamma(v_{n_0-1} \circ \mathcal{G}_1) + \Gamma(v_{n_0} \circ \mathcal{G}_1) \\
&\geq \cdots \geq \sum_{i\in[n_0]} \Gamma(\{v_i\} \circ \mathcal{G}_1) = n_0 \cdot \Gamma(\mathcal{G}_1) = \Gamma(\mathcal{G}_0) \cdot \Gamma(\mathcal{G}_1),
\end{aligned}$$

where the last equality follows from that $\Gamma(\mathcal{G}_0) = |V(\mathcal{G}_0)| = n_0$ since $\mathcal{G}_0$ is acyclic. ■

With help of Lemmas 4 and 5, we prove Theorem 4 below. We only provide proof for the induction base that (13) holds where $\mathcal{G}_0$ is at most 1-layer. Showing the inductive step can be done via similar steps to the proof of the induction base, utilizing the induction hypothesis instead of Lemma 5.

*Proof:* The proof for (13) when $\mathcal{G}_0$ is 0-layer is omitted. When $\mathcal{G}_0$ is 1-layer, assume that the disjoint acyclic chain $Ch \in \mathfrak{C}(\mathcal{G}_0)$, as shown in Definition 4, satisfies that $\Gamma(\mathcal{G}_0) = \Gamma(Ch)$. As $\mathcal{G}_0$ is an 1-layer problem, $Ch$ must be 1-layer with components being acyclic or half-rate-feasible. By Lemma 4, we can construct $Ch^{\mathcal{G}_1} \in \mathfrak{C}(\mathcal{G}_0 \circ \mathcal{G}_1)$ as:

$$\underline{I(1)\circ\mathcal{G}_1} \overset{\overset{K_{h_1}(1)\circ\mathcal{G}_1}{\cdots}}{\xleftrightarrow{K_1(1)\circ\mathcal{G}_1}}_{\mathrm{s}} \cdots \overset{\overset{K_{h_m}(m)\circ\mathcal{G}_1}{\cdots}}{\xleftrightarrow{K_1(m)\circ\mathcal{G}_1}}_{\mathrm{s}} \underline{I(m+1)\circ\mathcal{G}_1}.$$

Given that every component of $Ch$ is acyclic, we have

$$\begin{aligned}
&\Gamma(Ch^{\mathcal{G}_1}) \\
&= \frac{1}{m}(\sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(K_\ell(j)\circ\mathcal{G}_1) + \sum_{j\in[m+1]} \Gamma(I(j)\circ\mathcal{G}_1)) \\
&\geq \Gamma(\mathcal{G}_1) \cdot \frac{1}{m}(\sum_{j\in[m]}\sum_{\ell\in[h_j]} \Gamma(K_\ell(j)) + \sum_{j\in[m+1]} \Gamma(I(j))) \\
&= \Gamma(\mathcal{G}_1) \cdot \Gamma(Ch) = \Gamma(\mathcal{G}_1) \cdot \Gamma(\mathcal{G}_0), \quad (41)
\end{aligned}$$

where the inequality is due to Lemma 5, and the first and second equalities are due to Definition 5. Since $Ch^{\mathcal{G}_1} \in \mathfrak{C}(\mathcal{G}_0 \circ \mathcal{G}_1)$, by Definition 5, we also have

$$\Gamma(\mathcal{G}_0 \circ \mathcal{G}_1) = \max_{Ch' \in \mathfrak{C}(\mathcal{G}_0\circ\mathcal{G}_1)} \Gamma(Ch') \geq \Gamma(Ch^{\mathcal{G}_1}). \quad (42)$$

Combinning (41) and (42) leads to (13). ■

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