Abstract—This paper studies topology inference, from agent states, of a directed cyber-social network with opinion spreading dynamics model that explicitly takes confirmation bias into account. The cyber-social network comprises a set of partially connected directed network of agents at the social level, and a set of information sources at the cyber layer. The necessary and sufficient conditions for the existence of exact inference solution are characterized. A method for exact inference, when it is possible, of entire network topology as well as confirmation bias model parameters is proposed for the case where the bias mentioned earlier follows a piece-wise linear model. The particular case of no confirmation bias is analyzed in detail. Numerical simulations demonstrate the effectiveness of the proposed method.

I. INTRODUCTION

The inference of network topology, from observed state data, in dynamical systems is a key problem in several fields, ranging from bioinformatics [1] to communication [2] and social networks [3], see e.g., [4] for a comprehensive overview. We categorize the prior approaches to this problem into two groups: approximate and exact inference.

In several practical scenarios, the data to infer the network topology is only partially available or stochastic (e.g., noisy), which essentially renders network inference an instance of well-studied estimation problems with implicit assumptions on the dynamics. For example, in [5], this problem is studied in the context of structural equation models, while in [6], autoregressive models are employed. The solution approaches in the literature use tools from Bayesian analysis and estimation theory [7], [8], adaptive feedback control [9], [10], compressed sensing [11], or more generally optimization methods with sparsity constraints [12], [13].

In various other settings, an exact inference is possible. For example, in [14], a “node knockout” method is proposed where a subset of nodes are grounded (set to zero) to identify the network structure. In [15], this approach is coupled with power spectral analysis with the knowledge of eigenvalue-eigenvector of matrix that describes network structure. We note that while these methods provide exact topology inference, it is difficult in practice, if not impossible, to control the opinions of every individuals in a social network.

Perhaps closest to the proposed approach here, in [16], an exact topology inference strategy is presented, primarily for continuous-time consensus dynamics, by transforming the problem into a solution of Lyapunov equation whose numerical solutions are well studied, see e.g., the [17], [18]. This approach, unlike the ones in [5]–[10], [14], [15], does not require the capability of external stimulation for every node in the network, hence it is potentially applicable to social networks. However, as we demonstrate later in the text, this approach is not sufficient for exact inference in the directed network topologies that we consider throughout the paper.

We note that there exists a rich literature on dynamics of opinion evolution, see e.g., [19]–[21]. In [19], every individual updates her belief as a mean of her neighbors. In [20], the model further involves innate opinions, see also [22], [23]. In this paper, we focus on the topology inference problem of networks with known information dynamics such as the ones in [19]–[21], with the important explicit consideration of confirmation bias.

Confirmation bias of an individual refers to favoring information which confirms her previously existing beliefs. Confirmation bias has recently gained revived interest due to its role in the spread of misinformation, particularly due to its impact on creating an environment for misinformation to thrive in. It is well known that machine learning algorithms that filter news on social media feeds automatically utilize and foster this bias, see e.g., [24]–[26], hence contribute to the polarizations of public opinions. We consider this work as a part of comprehensive exploration of mathematical underpinnings of the misinformation spread in networks, see e.g., [22], [27]–[31].

In [32], opinion dynamics in social networks is studied with a particular focus on confirmation bias. Here, the cyber-social network comprises a social layer (individuals) and a cyber layer (information sources or namely stubborn individuals who do not change their opinions). The confirmation bias is modeled as a function of the distance between the opinions of individuals and information sources, explicitly taken into account in the dynamics model. We note that the well-known Hegselmann-Krause model [21] also addresses confirmation bias, where an individual completely ignores the opinions that are “too far” from hers. This model seems less amenable to detailed analysis than that in [32], which is adopted in this paper.

In this paper, building on the dynamics model in [32], we first investigate the necessary and sufficient conditions for solvability of exact inference of both network topology and bias parameters, in the case of piece-wise linear bias model with controlled information sources. We next develop a methodology, inspired by the approach in [16], to obtain exact inference in cases where it is possible. Next, we study the next inference question: if we ignore confirmation bias,
can we obtain exact inference of topology and bias, even without controlling the information sources?

II. Preliminaries

A. Notation

We let \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times n} \) denote the set of \( n \)-dimensional real vectors and the set of \( m \times n \)-dimensional real matrices, respectively. \( \mathbb{N} \) stands for the set of natural numbers, and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). We define \( I \) as the identity matrix with proper dimension. Moreover, we let \( 1 \) denote the vector of all ones.

The superscript \( \cdot^T \) stands for the matrix transposition. For a matrix \( W \in \mathbb{R}^{n \times n} \), \( [W]_{i,j} \) denotes the element in row \( i \) and column \( j \). \( A \in \mathbb{R}^{n \times n} \) is a row stochastic matrix if

\[
[A]_{i,j} \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} [A]_{i,j} = 1,
\]

for \( i, j = 1, \ldots, n \). Other important notations are highlighted as follows:

\[
\ker(Q): = \{ y : Qy = 0, Q \in \mathbb{R}^{n \times n} \}; \\
A^{-1}O: = \{ x : Ax \in O \}; \\
| \cdot |: = \text{(element-wise) modulus of a real (matrix) number.}
\]

The network considered in this paper is composed of \( n \) individuals (the social part of the network) and \( m \) information sources (the cyber part of the network). The interaction among the individuals is modeled by a digraph \( \Theta = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{ v_1, \ldots, v_n \} \) is the set of vertices representing the individuals and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the set of edges of the digraph \( \Theta \) representing the influence structure. Assume that the social network has no self-loops, i.e., for any \( v_i \in \mathcal{V} \), \( (v_i, v_i) \notin \mathcal{E} \).

The communication from information sources to individuals is modeled by a bipartite digraph \( \mathcal{H} = (\mathcal{V} \cup \mathcal{K}, \mathcal{E}) \), where \( \mathcal{K} = \{ k_1, \ldots, k_m \} \) is the set of vertices representing the information sources and \( \mathcal{B} \subseteq \mathcal{V} \times \mathcal{K} \) is the set of edges of the digraph.

B. Social Network Model

In this paper, we use the opinion dynamics in [32], [33]:

\[
x_i(k+1) = \alpha_i(x_i(k)) s_i + \sum_{j \in \mathcal{V}} w_{i,j} x_j(k) + \sum_{d \in \mathcal{K}} \hat{w}_{i,d}(x_i(k)) u_d
\]

where

1) \( x_i(k) \in [0, 1] \) is individual \( v_i \)'s opinion at time \( k \), \( s_i \) is her fixed innate opinion, \( u_d \in [0, 1] \) is the information source \( u_d \)'s opinion;
2) \( w_{i,j} \) represents the fixed weighted influence of individual \( v_j \) on individual \( v_i \),

\[
w_{i,j} \begin{cases} > 0, & \text{if} \ (v_i, v_j) \in \mathcal{E} \\ 0, & \text{otherwise}; \end{cases}
\]
3) \( \hat{w}_{i,d}(x_i(k)) \) is the weighted influence of information source \( u_d \) on individual \( v_i \) with

\[
\hat{w}_{i,d}(x_i(k)) = \begin{cases} g_i,d(|x_i(k) - u_d|), & \text{if} \ (v_i, u_d) \in \mathcal{B} \\ 0, & \text{otherwise}; \end{cases}
\]

where \( g_i,d : \mathbb{R} \rightarrow \mathbb{R} \) is a strictly decreasing function that models confirmation bias: an individual tends to seek out, and consequently be influenced more by, an information source who reflects beliefs closer to hers. We assume that \( g_i,d(s) \) satisfies \( 1 > g_i,d(|x_i(k) - u_d|) > 0 \).

4) \( \alpha_i(x_i(k)) \) is the “resistance parameter” of individual \( v_i \), determined in such a way that it satisfies

\[
\alpha_i(x_i(k)) + \sum_{j \in \mathcal{V}} w_{i,j} + \sum_{d \in \mathcal{K}} \hat{w}_{i,d}(x_i(k)) = 1, \quad i \in \mathcal{V}. \tag{3}
\]

Remark 1: We make a common assumption, similarly made in several related work, see e.g., [34], as the individual innate opinion is regarded as her initial opinion i.e.,

\[
s_i = x_i(0), \quad i \in \mathcal{V}. \tag{4}
\]

C. Related Prior Work

The most relevant prior work is the approach in [16], where an exact inference procedure of network topology for the continuous-time consensus dynamics with undirected communication. The discrete-time version version of consensus dynamics considered therein is

\[
\hat{x}(k+1) = \hat{L} \hat{x}(k),
\]

where \( \hat{L} \in \mathbb{R}^{n \times n} \) is a symmetric row stochastic matrix. The exact inference procedure of \( \hat{L} \) is based on the well-known numerical solutions of the constrained Lyapunov equation:

\[
\hat{L} U + U \hat{L}^T = V, \quad \hat{L} \in \mathbb{R}^{n \times n}, \quad \hat{L} 1 = 1, \quad [\hat{L}]_{i,j} \geq 0 \tag{5}
\]

where

\[
V \triangleq \sum_{k=0}^{m-1} (\hat{x}(k+1) \hat{x}^T(k) + \hat{x}(k) \hat{x}^T(k+1))
\]

and

\[
U \triangleq \sum_{k=0}^{m-1} \hat{x}(k) \hat{x}^T(k).
\]

However, this exact inference method works only for undirected communication topology, i.e., it cannot generate unique inference solution of directed communication topology when \( n \geq 4 \), which is demonstrated as follows,

In the context of directed communication, i.e., \( [\hat{L}]_{i,j} \neq [\hat{L}]_{j,i} \), the relation (5) would update as

\[
\hat{L} U + U \hat{L}^T = V, \quad \hat{L} \in \mathbb{R}^{n \times n}, \quad \hat{L} 1 = 1, \quad [\hat{L}]_{i,j} \geq 0. \tag{6}
\]

For the directed communication graph with \( n \) agents, there are \( n^2 - n \) weighted communication links (variables) \([\hat{L}]_{i,j}, i \neq j \), need to be inferred. In (6), both matrices \( \hat{L} U + U \hat{L}^T \) and \( V \) are symmetric. Thus, (6) contains, at most, \( \frac{n(n-1)}{2} \) distinct linear equations that are related to one or some \([\hat{L}]_{i,j}, i \neq j \). Moreover, we note that the constraint conditions \( \hat{L} 1 = 1 \) and \([\hat{L}]_{i,j} \geq 0 \) in (6) only reduce the number of linear equations while do not affect the number of variables to be inferred. It can be verified that \( n^2 - n - \frac{(n-1)n}{2} = \frac{(n-3)n}{2} > 0 \) when \( n > 4 \), which indicates when the network includes more than four agents, the number
of inferring weighted links is larger than the number of equations included in (6). Therefore, we conclude that the exact inference procedure based on constrained Lyapunov equation cannot be applied to the directed communication graph when \( n > 4 \).

In the following sections, we present a novel inference procedure for networks with directed communication.

D. Problem Formulation

In this paper, we first investigate on the inference problem for the case where all information sources are under control and the individual confirmation bias follows a linear model, i.e., the function \( g_{i,d}(\cdot) \) in (2) is described by

\[
g_{i,d}(x_i(k)) = \beta_i - \gamma_i [x_i(k)] - u_d. \tag{7}
\]

We next study inference problem in the case where no individual holds confirmation bias and the opinions of information sources are not under control. The studied problems in the two different cases are formally stated as follows.

**Inference Problem I:** For the opinion evolution with confirmation bias \( (\gamma_i \neq 0 \text{ for some } i \in \mathcal{V}) \), given controlled opinions of information sources and measured evolving opinions \( x(k) \) at time \( k = 1, \ldots, m \), exactly infer the network topology and the confirmation bias.

**Inference Problem II:** For the opinion evolution without confirmation bias \( (\gamma_i = 0 \text{ for any } i \in \mathcal{V}) \), given uncontrolled opinions of information sources and measured evolving opinions \( x(k) \) at time \( k = 1, \ldots, m \), exactly infer social network topology.

**Remark 2:** In order to obtain the exact inference, we need a “global capability”: measure all of the individuals’ evolving opinions for some time periods. As also mentioned in Remark 2 of [16], such a global capability is necessary in the exact topology reconstruction with consensus-seeking dynamics.

III. INFEERENCE PROBLEM I

In Problem I, every information source’s opinion is a feasible control variable. To simplify the inference procedure, we can control all of the information sources to express the same “zero” opinion:

\[
u_1 = u_2 = \ldots = u_m = u = 0. \tag{8}
\]

Since the information sources express the same opinion, we mathematically treat them as one information source \( u \). It follows from the convex combination (3) and the social dynamics (1) that \( x_i(k) \in [0,1], \forall i \in \mathcal{V}, \forall k \in \mathbb{N} \). Therefore, the state-dependent weights (2) under the opinion (8) and the linear model (7) satisfy

\[
\hat{w}_{i,d}(x_i(k)) = \beta_i - \gamma_i x_i(k). \tag{9}
\]

We note here that the communication from information source \( u \) to individuals is incorporated into the parameters \( \beta_i \) and \( \gamma_i \):

\[
\beta_i \begin{cases} > 0, & (v_i, u) \in \mathbb{B}, \\ = 0, & \text{otherwise}, \end{cases} \quad \gamma_i \begin{cases} > 0, & (v_i, u) \in \mathbb{B}, \\ = 0, & \text{otherwise}, \end{cases}
\]

Consequently, the resistance parameters are obtained from (3) as

\[
\alpha_i(x_i(k)) = 1 - \sum_{j \in \mathcal{V}} w_{i,j} - \beta_i + \gamma_i x_i(k), \quad i \in \mathcal{V}. \tag{10}
\]

Under the settings of (4) and (8), it follows from (9) and (10) that the social dynamics (1) equivalently expresses as

\[
x(k + 1) = Ax(0) + Wx(k), \tag{11}
\]

where we define:

\[
A \triangleq \text{diag}\{1 - \sum_{j \in \mathcal{V}} w_{1,j} - \beta_1, \ldots, 1 - \sum_{j \in \mathcal{V}} w_{n,j} - \beta_n\},
\]

\[
[W]_{i,j} \triangleq \begin{cases} \gamma_i x_i(0), & i = j \in \mathcal{V} \\ w_{i,j}, & i \neq j \in \mathcal{V}. \end{cases} \tag{12}
\]

A. Solvability of Inference Problem I

We now consider the following dynamics

\[
x(k + 1) = \widehat{A}x(0) + \widehat{W}x(k), \tag{13}
\]

where \( \widehat{A} \) and \( \widehat{W} \) as referred to the inferred matrices for \( A \) and \( W \), respectively. Given the measured evolving opinions \( x(k), k = 0, 1, \ldots, m \), the solvability of exact inference problem of network topology and confirmation bias (in term of its parameters \( \beta_i \) and \( \gamma_i \)) must satisfy the following two conditions:

C1: \( x(k) \) is evolving opinion to both the social dynamics (11) and (13) for time \( k = 1, \ldots, m \).

C2: \( \widehat{W} = W, \beta_i = \beta_i \) and \( \gamma_i = \gamma_i, i \in \mathcal{V} \).

**Remark 3:** If C2 requires \( \widehat{A} = A \) and \( \widehat{W} = W \) alternatively, by (12) we can only uniquely infer the weighted social network topology: \( w_{i,j} = [W]_{i,j}, i \neq j \in \mathcal{V} \). However, the inference of confirmation bias parameters is not necessarily unique, which can be verified by \( [W]_{i,i} = \gamma_i x_i(0) = \widehat{[W]}_{i,i}, i \in \mathcal{V} \), when \( x_i(0) = 0 \).

This subsection investigates on the conditions of inference solvability, which will contribute to deriving the exact inference procedures.

**Proposition 1:** [35] \( x(k) \) is evolving opinion to both the social dynamics (11) and social dynamics (13) for \( k = 1, \ldots, m \), if and only if

\[
x(0) \in L^{-1} \ker(\widehat{O}) \cap \ker(\widehat{A} + \widehat{W}),
\]

where

\[
L \triangleq A + W - I,
\]

\[
\widehat{A} \triangleq \widehat{A} - A, \quad \widehat{W} \triangleq \widehat{W} - W,
\]

\[
\widehat{O} \triangleq \begin{bmatrix} \widehat{W}^\top, (\widehat{WW})^\top, \ldots, (\widehat{WW}^{m-1})^\top \end{bmatrix}^\top.
\]

Based on Proposition 1, we directly obtain the sufficient and necessary condition on the solvability of the inference problem.

**Corollary 1:** The inference of network topology and confirmation bias is solvable for the social dynamics (11), if and
only if for any \( w_{i,j} \neq \tilde{w}_{i,j}, i \neq j \in V \), or \( \beta_i \neq \tilde{\beta}_i, i \in V \), or \( \gamma_i \neq \tilde{\gamma}_i, i \in V \), such that

\[
x(0) \notin L^{-1} \ker(\hat{O}) \cap \ker(\hat{A} + \hat{W}).
\]

The sufficient and necessary condition obtained in Corollary 1 requires the knowledge of inference errors of encoded matrices, i.e., \( \hat{A} \) and \( \hat{W} \). In the following, we provide a sufficient condition of inference solvability that does not have such requirement.

**Theorem 1:** [35] The exact inference problem for the social dynamics (11) is solvable if

\[
x_i(0) \neq 0, \quad \forall (v_i, u) \in \mathbb{B}, \quad \text{rank} \left( [Lx(0), W Lx(0), \ldots, W^{n-1} Lx(0)] \right) = n.
\]

Although the obtained sufficient conditions in Theorem 1 guarantee a unique inference solution, we cannot use the relations (15) and (14) since the matrices \( W \) and \( L \) are unavailable. In the following, we develop an exact inference procedure, based on Theorem 1, without using \( W \) and \( L \).

**B. Exact Solution to Inference Problem I**

To derive the inference solution, let us consider the following matrix that makes use of the available measurement of evolving opinions:

\[
P \triangleq \sum_{k=0}^{m-1} (x(k+1) - x(k))(x(k+1) - x(k))^\top, \quad m \in \mathbb{N}.
\]

**Remark 4:** To exactly infer the discrete-time dynamical network with undirected topology, Waarde et al. [16] suggest a usage of available collected opinions:

\[
P \triangleq \sum_{t=0}^{m-1} x(k)x^\top(k).
\]

Using the same derivation method for inference procedures in [16], it follows the dynamics (11) and the function (17) that

\[
WP = \sum_{t=0}^{m-1} W x(k)x^\top(k) = \sum_{k=0}^{m-1} (x(k) - Ax(0))x^\top(k)
\]

\[
= \sum_{k=0}^{m-1} x(k+1)x^\top(k) - \sum_{k=0}^{m-1} Ax(0)x^\top(k).
\]

Due to the unavailable encoded matrix \( A \), (18) indicates the suggested function (17) cannot be used for the dynamics of opinion evolution in the presence of confirmation bias (11) in deriving inference procedures.

The following auxiliary lemmas present some properties of \( P \), which will be used in deriving inference procedures.

**Lemma 1:** [35] Consider the matrix (16). For the social dynamics (11), we have

\[
WP = Q,
\]

where

\[
Q \triangleq \sum_{k=0}^{m-1} (x(k+2) - x(k+1))(x(k+1) - x(k))^\top.
\]

**Lemma 2:** [35] Consider the matrix (16). For the social dynamics (11), we have

\[
\ker(P) = \ker([Lx(0), W Lx(0), \ldots, W^{m-1} Lx(0)])^\top.
\]

In the relation (19), the defined matrix \( P \) and \( Q \) are known since they are computed from the available measurements \( x(k), k = 0, 1, m - 1 \). If the encoded matrix \( W \) can be uniquely obtained from (19), we can uniquely infer the fixed influence weights \( w_{i,j} = [W]_{i,j} \), with \( i \neq j \), which as well describe the social network topology \((v_i, v_j) \in \mathbb{E} \) if \( w_{i,j} \neq 0 \), and the bias parameters \( \gamma_i = \frac{[W]_{i,i}}{x_i(0)} \) that contain the information of communication from information source \( u \) to individuals \(((v_i, u) \in \mathbb{B} \) if \( \gamma_i \neq 0 \). With the obtained \( W \), the left parameters \( \beta_i \) can be obtained via considering social dynamics.

**Corollary 2:** [35] Consider social dynamics (11). Given \( W, x(0), x(k) \) and \( x(k+1) \), the parameters \( \beta_i \) of confirmation bias are solved as

\[
\beta_i = 1 - \sum_{j \neq i \in V} [W]_{i,j} x_j(k) x_i(0) .
\]

In the following theorems, we present our results on the exact inference problem.

**Theorem 2:** [35] The exact inference problem of network topology and confirmation bias is solvable for the social dynamics (11), if and only if (14) holds and there exists a unique \( \tilde{W} \in \mathbb{R}_n \times n \) such that

\[
\tilde{W}P = Q,
\]

where \( P \) and \( Q \) are defined in (16) and (20), respectively.

The solvability of inference problem should be checked before computing (22). However, the solvability condition obtained in Theorem 1 requires the knowledge of \( L = W + A - I \) that is unavailable. The following theorem makes use of the available \( P \) instead of \( L \) to check the solvability, and then solve the inference problem.

**Theorem 3:** [35] Consider the matrices \( P \) and \( Q \) given by (16) and (20), respectively. If the condition (14) holds and rank \((P) = n \), the network topology and bias parameters (11) are exactly inferred as

\[
\begin{align*}
\gamma_i &= \frac{[Q P^{-1}]_{i,i}}{x_i(0)}, \quad i \in V \\
\beta_i &= 1 - \sum_{j \neq i \in V} [Q P^{-1}]_{i,j} x_j(k) x_i(0),
\end{align*}
\]

where

\[
\begin{align*}
\gamma_i &= \frac{[Q P^{-1}]_{i,i}}{x_i(0)}, \quad i \in V \\
\beta_i &= 1 - \sum_{j \neq i \in V} [Q P^{-1}]_{i,j} x_j(k) x_i(0), \quad i \in V.
\end{align*}
\]
Remark 5: Using the inferred matrices $W$ and $A$ and the available data $x(0)$, the steady state of evolving opinions is exactly inferred from $Ax(0) + Wx^* = x^*$:

$$x^* = (I - W)^{-1}Ax(0).$$

IV. Inference Problem II

In the scenario with no confirmation bias described by (7) $\gamma_i = 0, \forall i \in V$, if the opinions of information sources are still controlled, (23) and (25) straightforwardly generate the exact inference. Hence, in this scenario, we investigate the question of whether the topology can still be exactly inferred even if the opinions of information sources are not under control.

We now consider the dynamics that is slightly modified from (7)

$$x_i(k+1) = \alpha_i s_i + \sum_{j \in V} w_{i,j} x_j(k) + \sum_{d \in \mathbb{K}} \hat{w}_{i,d} u_d, i \in V$$

(26)

where $\hat{w}_{i,d}$ represents the fixed weighted influence of information source $u_d$ on individual $v_i$, the fixed resistance parameter $\alpha_i$ of individual $v_i$ is determined in such a way that it satisfies

$$\alpha_i + \sum_{j \in V} w_{i,j} + \sum_{d \in \mathbb{K}} \hat{w}_{i,d} = 1, \forall i \in V.$$

(27)

Under the setting of (4), it follows from (27) that (26) equivalently can be expressed as the following.

$$x(k+1) = Ax(0) + Wx(k),$$

(28)

where we define

$$A \triangleq \text{diag}\left\{1 - \sum_{j \in V} w_{i,j} - \sum_{d \in \mathbb{K}} \hat{w}_{i,d} + \sum_{d \in \mathbb{K}} \frac{\hat{w}_{i,d} u_d}{x_1(0)} , \ldots, 1 - \sum_{j \in V} w_{n,j} - \sum_{d \in \mathbb{K}} \hat{w}_{n,d} + \sum_{d \in \mathbb{K}} \frac{\hat{w}_{n,d} u_d}{x_n(0)}\right\},$$

$$\left[\mathcal{W}\right]_{i,j} \triangleq \begin{cases} 0, & i = j \in V \\ w_{i,j}, & i \neq j \in V. \end{cases}$$

(29)

We note that the dynamics (11) and (28) have the same form. Therefore, the analysis method in deriving the inference procedure for (11) can be employed for (28).

Corollary 3: Consider the matrices $P$ and $Q$ given by (16) and (20), respectively. If rank $(P) = n$, the social network topology of dynamics (28) is exactly inferred as

$$\mathcal{W} = QP^{-1}.$$

(31)

Remark 6: The dynamics (28) results in

$$\sum_{d \in \mathbb{K}} \hat{w}_{i,d} (u_d - x_i(0))$$

(32)

$$= x_i(k+1) - \sum_{j \in V} [\mathcal{W}]_{i,j} x_j(k) - (1 - \sum_{j \in V} [\mathcal{W}]_{i,j}) x_i(0),$$

whose right-hand side is known, since the evolving and innate opinions $x_i(k)$ and $x_i(0), i \in V$, are available measurement data, and $\mathcal{W}$ is obtained from (31). When the social network is in the presence of several information sources, i.e., $|\mathbb{K}| \geq 2$, the left-hand side of (32) has more than one variables $\hat{w}_{i,d}, d \in \mathbb{K}$, to be inferred from the single equation. Therefore, we conclude that the topology from information sources to individuals, described by $\hat{w}_{i,d}$, cannot be uniquely inferred, even if individual does not hold confirmation bias.

V. Simulations

In this section, we numerically simulate the network with $n = 12$ individuals, whose network topology with coupling weights and bias parameters are shown in Figure 1 (a). The innate opinions are randomly generated as $x(0) = [0.7513, 0.2551, 0.506, 0.6991, 0.8909, 0.9593, 0.5472, 0.1386, 0.1493, 0.2575, 0.8407, 0.2543]^T$.

For the confirmation bias follows from linear model (7), this subsection verifies the exact inference. We observe and collect the data of involving opinions at times $k = 1, 2, \ldots, 12$. Using the collected data, we construct matrices $P$ and $Q$ via (16) and (20), respectively. It can be verified that $P$ is full-rank, and it obtains from (22) that

$$\bar{W}_{1,1} = 0.2254, \bar{W}_{2,2} = 0.0510, \bar{W}_{3,3} = 0.0506, \bar{W}_{4,4} = 0.0699, \bar{W}_{1,12} = 0.4, \bar{W}_{2,1} = 0.5, \bar{W}_{3,2} = 0.6, \bar{W}_{4,3} = 0.7, \bar{W}_{5,4} = 0.1, \bar{W}_{5,7} = 0.2, \bar{W}_{5,10} = 0.3, \bar{W}_{6,5} = 0.2, \bar{W}_{6,7} = 0.3, \bar{W}_{7,6} = 0.5, \bar{W}_{7,11} = 0.2, \bar{W}_{8,7} = 0.1, \bar{W}_{8,10} = 0.7, \bar{W}_{9,8} = 0.8, [\bar{W}]_{10,9} = 0.6, [\bar{W}]_{11,10} = 0.9, [\bar{W}]_{12,6} = 0.2, [\bar{W}]_{12,11} = 0.5,$$

other $\hat{w}_{i,j}$’s are zeros. Through comparing the nonzero weights in (33) with the coupling weights $w_{i,j}, i \neq j$, in the social network in Figure 1, we conclude that the
inference procedure in Theorem 3 generates exact inference of weighted social network topology. Moreover, since $x_i(0) \neq 0$ for $i = 1, 2, 3, 4$. By (24) and $[W]_{i,i}, i = 1, 2, 3, 4, \in (33)$, we have $\gamma_1 = \frac{0.2254}{0.7513} = 0.3, \gamma_2 = \frac{0.0510}{0.2551} = 0.2, \gamma_3 = \frac{0.0006}{0.0006} = 1, \gamma_4 = \frac{0.6998}{0.6998} = 1$. Thus, the parameters $\gamma_i$ of the followers’ confirmation bias model are exactly inferred. Finally, the remaining parameters are obtained from (25): $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.5, 0.4, 0.3, 0.2)$. These computation results with Figure 1 demonstrate the effectiveness of exact inference procedure presented in Theorem 3.

VI. CONCLUSION

In this paper, we have studied the inference problems in the cyber-social networks with confirmation bias. In the case where the individual confirmation bias follows a piece-wise linear model, we have derived an exact inference procedure of network topology and confirmation bias. Numerical simulations demonstrate the effectiveness of exact inference procedure.

As a part of future work, we will study the impact of an adversary (or competitor) controlling a subset of information sources on the topology inference. Such consideration is expected to bring a trade-off between the performances of the topology inference and controlling opinion evolution.

REFERENCES


