Serial Quantization for Representing Sparse Signals

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Abstract—Sparse signals are encountered in a broad range of applications. In order to process these signals using digital hardware, they must be first quantized using an analog-to-digital converter (ADC), which typically operates in a serial scalar manner. In this work we propose a method for serial quantization of sparse signals (SeQuanS) inspired by group testing theory, which is designed to reliably and accurately quantize sparse signals acquired in a sequential manner using serial scalar ADCs. Unlike previously proposed approaches which combine quantization and compressed sensing (CS), our SeQuanS scheme updates its representation on each incoming analog sample and does not require the complete set of analog signal samples to be observed and stored in analog prior to quantization. We characterize the asymptotic tradeoff between accuracy and quantization rate of SeQuanS as well as its computational burden. Our numerical results demonstrate that SeQuanS is capable of achieving substantially improved representation accuracy over previous CS-based schemes without requiring the complete set of analog signal samples to be observed prior to its quantization, making it an attractive approach for acquiring sparse time sequences.

I. INTRODUCTION

Quantization allows continuous-amplitude physical signals to be represented using discrete values and processed in digital hardware. Such continuous-to-discrete conversions play an important role in digital signal processing systems [1]. In theory, jointly mapping a set of samples via vector quantization yields the most accurate digital representation [2, Ch. 10]. However, as such joint mappings are difficult to implement, quantization is most commonly carried out using analog-to-digital converters (ADCs), which operate in a serial and scalar manner, namely, each incoming sample is sequentially mapped into a discrete representation using the same mapping [3]. Since ADCs operating at high frequencies are costly in terms of memory and power usage, it is often desirable to utilize lower quantization rates, i.e., assign a limited number of bits per each input sample, inducing additional quantization error which degrades the digital representation accuracy [4, Ch. 23].

The quantization error encountered under bit budget constraints can be mitigated by accounting for underlying structure or the system task. Such quantization systems are the focus of several recent works. For example, scalar quantization mappings designed to maximize the mutual information and Fisher information with respect to a statistically related quantity were studied in [5] and [6], respectively. The work [7] showed that a quantization system using uniform ADCs can approach the performance achievable using vector quantizers when the system task is not to recover the analog signal, but to estimate some lower-dimensional information embedded into it. This approach was extended to massive multiple-input multiple-output (MIMO) channel estimation with quantized outputs in [8] as well as to the recovery of quadratic functions in [9]. The systems proposed in [7]–[9] used hybrid architectures, namely, allowed some constrained processing to be carried out in analog prior to quantization, in order to mitigate the error induced by bit-limited serial scalar ADCs.

A common structure exhibited by physical signals is sparsity. Sparse signals are frequently encountered in various applications, ranging from biomedical and optical imaging [10], [11] to radar [12] and communications [13], [14]. An important property of sparse signals is the fact that they can be perfectly reconstructed from a lower-dimensional projection without knowledge of the sparsity pattern. This property is studied within the framework of compressed sensing (CS) [15], [16], which considers the design and analysis of algorithms for recovering sparse signals from their lower-dimensional projections.

Recovery of sparse signals from quantized measurements is the focus of a large body of work [17]–[23]. The most common approach studied in the literature is to first project the signal in the analog domain and then quantize the compressed measurements, via one bit representation [17], [18], uniform quantization [19], sigma-delta quantization [20], [23], or vector source coding [21]. A detailed survey and analysis of methods combining quantization and CS can be found in [22]. The aforementioned approaches all require the sparse signal to be projected prior to quantization. This requirement constitutes a major drawback when the sparse signal represents a time sequence whose samples are acquired sequentially, which is the case in many signal processing and communication applications, as the entire sparse signal must be first fully obtained and stored in analog in order to project and quantize it. Furthermore, while CS algorithms have been proven to achieve asymptotic recovery guarantees, their performance may be degraded in finite signal sizes. These drawbacks give rise to the need for a reliable and sequential method for quantizing and recovering sparse signals, which is the focus of this work.

Here, we propose serial quantization of sparse signals (SeQuanS), which is a method for quantizing and recovering sparse signals operating in a sequential manner and utilizing standard serial scalar ADCs. Our scheme is inspired by recent developments in group testing theory, combining coding principles designed for secure group testing [24], with typical hardware limitations of digital signal processors and conventional ADCs. The proposed SeQuanS system is capable of reliably operating under strict bit budgets without storing samples in analog by sequentially updating a single register.
used for digital representation.

We characterize the achievable accuracy of SeQuanS in the asymptotically large signal size regime, showing that any fixed desirable distortion level can be achieved with an overall number of bits which grows logarithmically in the signal dimensionality and linearly with the support size. Our numerical results demonstrate that SeQuanS substantially outperform the conventional approach combining CS and quantization for finite signal sizes. This demonstrates the potential of SeQuanS for feasible and reliable quantization of sequentially acquired sparse signals.

The rest of this paper is organized as follows: In Section II we review some preliminaries in quantization theory and present the system model. Section III proposes SeQuanS along with a discussion and an asymptotic performance analysis. Section IV details the simulation study, and Section V provides concluding remarks. Proofs of the results stated in the paper are detailed in the appendix.

Throughout the paper, we use boldface lower-case letters for vectors, e.g., $\mathbf{x}$; the $i$th element of $\mathbf{x}$ is written as $x_i$. Matrices are denoted with boldface upper-case letters, e.g., $\mathbf{M}$. Sets are expressed with calligraphic letters, e.g., $\mathcal{X}$, and $\mathcal{A}^n$ is the $n$th order Cartesian power of $\mathcal{X}$. The stochastic expectation is denoted by $\mathbb{E}\{\cdot\}$, the Boolean OR operation is expressed as $\lor$, while $\mathbb{R}$ is the set of real numbers. All logarithms are taken to base-2.

II. PRELIMINARIES AND SYSTEM MODEL

A. Preliminaries in Quantization Theory

To formulate the quantization of sparse signals setup, we first briefly review standard quantization notions. We begin with the definition of a quantizer:

**Definition 1** (Quantizer). A quantizer $Q^n_M(\cdot)$ with $\log M$ bits, input size $n$, input alphabet $\mathcal{S}$, output size $m$, and output alphabet $\mathcal{S}_m$ consists of: 1) An encoding function $g^n_m : \mathcal{S}^n \mapsto \{0, 1, \ldots, M - 1\} \triangleq \mathcal{M}$ which maps the input from $\mathcal{S}^n$ to a discrete index $i \in \mathcal{M}$. 2) A decoding function $g^m_i : \mathcal{M} \mapsto \mathcal{S}^m$ which maps each index $i \in \mathcal{M}$ onto a codeword $q_i \in \mathcal{S}^m$.

The quantizer output for an input $s = \{s_i\}_{i=1}^n \in \mathcal{S}^n$ is $\hat{s} = g^m_i(g^n_m(s)) \triangleq Q^n_M(m)$. An illustration is depicted in Fig. 1. Scalar quantizers operate on a scalar input, i.e., $n = 1$ and $\mathcal{S}$ is a scalar space, while vector quantizers have a multivariate input. The set $\{q_i\}_{i=1}^M$ is referred to as the quantization codebook. When the input size and output size are equal, namely, $n = m$, we write $Q^n_m(\cdot) \triangleq Q^n_M(\cdot)$.

In the standard quantization problem, a $Q^n_M(\cdot)$ quantizer is designed to minimize some distortion measure $d_n : \mathcal{S}^n \times \hat{\mathcal{S}}^n \mapsto \mathbb{R}^+$ between its input and its output. The performance of a quantizer is therefore characterized using two measures: The quantization rate, defined as $R \triangleq \frac{1}{n} \log M$, and the expected distortion $\mathbb{E}\{d_n(s, \hat{s})\}$. For a fixed input size $n$ and codebook size $M$, the optimal quantizer is given by

$$Q^n_M(\cdot) = \min_{Q^n_M(\cdot)} \mathbb{E}\{d_n(s, Q^n_M(s))\}.$$  

(1)

In the following, the distortion between a source realization $s$ and a reconstruction sequence $\hat{s}$ is defined as the mean-squared error (MSE) of their difference given by

$$d_n(s, \hat{s}) \triangleq \frac{1}{n} ||s - \hat{s}||^2 = \frac{1}{n} \sum_{i=1}^n (s_i - \hat{s}_i)^2.$$  

(2)

Characterizing the optimal quantizer via (1) and the distortion via (2), as well as the optimal tradeoff between distortion and quantization rate, is in general a very difficult task. Consequently, optimal quantizers are typically studied assuming either high quantization rate, i.e., $R \rightarrow \infty$, see, e.g., [25], or asymptotically large input size, namely, $n \rightarrow \infty$, typically with stationary inputs, via rate-distortion theory [26, Ch. 10].

Comparing high rate analysis for scalar quantizers and rate-distortion theory for vector quantizers demonstrates the sub-optimality of serial scalar quantization. For example, for quantizing a large-scale real-valued Gaussian random vector with i.i.d. entries and sufficiently large quantization rate $R$, where intuitively there is little benefit in quantizing the entries jointly over quantizing each entry independently, vector quantization notably outperforms serial scalar quantization [4, Ch. 23.2]. Nonetheless, vector quantizers are significantly more complex compared to serial scalar quantizers. One of the main sources for this increased complexity stems from the fact that vector quantizers operate on a set of analog samples. As a result, a digital signal processor (DSP) utilizing vector quantizers to acquire a physical signal must store a set of $n$ samples in the analog domain before it can produce a digital representation, which may be difficult to implement, especially for large $n$. Scalar quantizers do not require storing data in analog as each incoming sample is immediately converted into a digital representation.

B. System Model

We consider the acquisition of an $n$-dimensional signal $s \in \mathbb{R}^n$ into a digital representation $\hat{s} \in \mathcal{R}^n$ using up to $b$ bits, i.e., $M = 2^b$ codewords. The performance is measured by the MSE distortion $\mathbb{E}\{|s - \hat{s}|^2\}$ and the quantization rate $R = \frac{b}{n}$. The signal $s$ is assumed to be sparse with support size $k \ll n$, where $k$ is a-priori known. We propose a quantization system which is specifically designed to exploit this sparsity to improve the recovery accuracy. In particular, we propose an encoder-decoder pair which utilizes tools from group testing theory to exploit the underlying sparsity of the continuous amplitude signal.

We focus the on the scenario in which $s$ represents $n$ consecutive samples taken from some analog signal, namely, the $i$th entry of $s$, denoted $s_i$, $i \in \{1, \ldots, n\} \triangleq \mathcal{N}$, represents

\footnote{If $k$ is not known a-priori, we can use the methods and bounds given in [27], [28] to learn the value of $k$ with $\mathcal{O}(\log n)$ outcome bits.}
a sample acquired at discrete-time index \( i \). Such scenarios represent serial ADCs typically utilized by DSPs [29]. In order to avoid the need to store samples in analog, the system operates on each entry of \( s \) independently. In particular, on each incoming sample \( s_i \), the encoder updates a register of \( b \) bits, whose value upon the encoding of \( s_i \) is denoted by \( y_i \). Once the complete vector \( s \) is acquired, the decoder uses the digital codeword \( y_n \) to produce an estimate of \( s \) denoted \( \hat{s} \in \mathbb{R}^n \). An illustration of the system is depicted in Fig. 2. Since the decoder process discrete codeword \( y_n \), while each \( y_i, i < n \) is stored only during the \( i \)th acquisition step, the system uses \( b \) bits for digital representation.

### III. SeQuaNS System

We next detail the proposed SeQuanS system. The main rationale of SeQuanS is to facilitate quantization of sparse signals using conventional low-complexity serial scalar quantizers by utilizing group theory tools. Broadly speaking SeQuanS quantizes each incoming sample using a scalar ADC. However, instead of storing this quantized value, it is used to update a \( b \) bits codeword, which is decoded into a digital representation of the sparse signal. This approach allows to quantize each incoming samples with relatively high resolution, while using a single register of \( b \) from which the digital representation of the complete signal is obtained. To properly formulate SeQuanS, we first present the codebook generation in Subsection III-A. Then, we elaborate on the SeQuanS encoder and decoder structures in Subsection III-B and III-C, respectively. In Subsection III-D we characterize the achievable distortion of SeQuanS in the large signal size regime. Finally, in Subsection III-E we discuss the strengths and weaknesses of SeQuanS compared to previously proposed approaches for quantizing sparse signals.

#### A. Codebook Generation

The SeQuanS system maintains a codebook used by its encoder and decoder. In particular, for an input signal of size \( n \), SeQuanS uses a codebook of \( l \cdot n + 1 \) codewords, each consisting of \( b \) bits, where \( l \) is a fixed integer. We discuss the effect of \( l \) on the MSE and the complexity of SeQuanS in Subsection III-D, and propose guidelines for determining its value to optimize the tradeoff between these key performance measures.

Our codebook design is based on the codebook given in [30] for wireless sensor networks which is inspired by recent advances in group testing theory [31], and particularly the code proposed in [32] for secure group testing. The objective in group testing is to identify a subset of defective items in a larger set using as few measurements as possible. This objective can be recast as a codebook generation problem, such that for each outcome vector, i.e., a set of measurements, it should be possible to identify the inputs that are not zero [31]. While this setup bears much similarity to our quantization of sparse sources problem, in group testing the inputs are represented over a binary field, while in our setting the inputs can be any real value. Consequently, the codebook here needs to be able not only to detect the indexes of those non zero inputs, as in conventional group testing, but also to recover their value. To facilitate our design, we henceforth assume that the inputs are discretized to a set of \( l + 1 \) different values, and show how this is incorporated into the overall encoder-decoder scheme in the following subsections.

In particular, to generate the codebook, we generate \( n \cdot l \cdot b \) independent realizations from a Bernoulli distribution with mean value \( \frac{\ln(2)}{2} \). These realizations form \( l \cdot n \) mutually independent codewords. The codewords are then divided into \( n \) bins, denoted \( B_i \triangleq \{ q_{ij} \}_{j=1}^l, i \in \mathcal{N} \), and we add to each bin the all-zero codeword denoted \( c_0 \). Since \( c_0 \) is common to all the bins, the total number of codewords is \( l \cdot n + 1 \). The benefits of this codebook design are discussed in Subsection III-E.

#### B. Encoder Structure

Having generated \( n \) bins of \( l \) codewords, \( \{ B_i \}_{i=1}^n \), we now discuss the encoding process. To that aim, we fix some scalar quantization mapping over \( \mathcal{R} \) with resolution \( l + 1 \), denoted \( Q_{l+1}(\cdot) \), and let \( \{ q_j \}_{j=0}^l \) be the set of its possible outputs. The specific selection of the quantization mapping represents the acquisition hardware. For example, when using the common flash ADC architecture, \( Q_{l+1}(\cdot) \) represents a uniform quantization mapping with \( l + 1 \) uniformly spaced decision regions [3]. Without loss of generality, we assume that the scalar quantizer maps the input value 0 into the discrete value \( q_0 \), namely, \( Q_{l+1}(0) = q_0 \).

The encoding process consists of the following three stages, illustrated in Fig. 3:

1. Each incoming sample \( s_i \) is quantized into the discrete scalar value \( Q_{l+1}(s_i) \). Since this same identical mapping is applied to each incoming sample in a serial manner, it can be implemented using conventional serial scalar ADCs.
2. The encoder uses the index of the discrete value \( Q_{l+1}(s_i) \) to select a codeword from the \( i \)th bin via the following assignment: If \( q_j = Q_{l+1}(s_i) \), then the selected codeword is \( \hat{c}_i = c_{j,i} \in B_i \).
3. The encoder output \( y_i \), which is initialized such that \( y_0 \) is the all-zero vector, is updated by taking its Boolean OR with the selected codeword \( \hat{c}_i \), i.e.,

\[
y_i = y_{i-1} \vee \hat{c}_i.
\]

Consequently, the encoder output \( y_n \) is given by

\[
y_n = \bigvee_{i=1}^n \hat{c}_i.
\]
Figure 3: Encoding process of the SeQuanS system.

Note that only the discrete index of the quantized $Q_{l+1}^1(s_i)$, and not its actual value, affects the selection of the encoder output $y_n$. Nonetheless, in Subsection III-D we show that the selection of the output of $Q_{l+1}^1(\cdot)$, i.e., the values of $\{q_j\}$, and not only its partition of $\mathcal{R}$ into decision regions, affect the overall MSE of the SeQuanS system. Additionally, the formulation of the encoder output via (3) implies that it can be represented using a single register of $b$ bits, which is updated using basic logical operations on each incoming sample. Consequently, while the encoding process assigns a $b$ bits codeword to each incoming sample, the overall output size is $b$ and not $n \cdot b$, thus the quantization rate is $R = \frac{b}{n}$.

C. Decoder Structure

The recovery of the digital representation $\hat{s} \in \mathbb{R}^n$ from the output of the encoder $y_n \in \{0, 1\}^b$ is based on maximum likelihood (ML) decoding. In this decoding scheme, the most likely set of $k$ codewords are selected, from which the digital representation is obtained. To formulate the decoding process, recall that the set $\mathcal{N}$ has exactly $\binom{n}{k}$ possible subsets of size $k$, representing the possible sets of non-zero entries of $s$. We use $\{X_w \}_{w \in \{1, \ldots, \binom{n}{k}\}}$ to denote these subsets. The SeQuanS decoder implements the following steps:

- For a given encoder output $y_n$, the decoder recovers a collection of $k$ codewords $\hat{C}_{X_w} = \{c_{j_i,w}\}_{i \in X_w}$, each one taken from a separate bin, for which $y_n$ is most likely, namely,
  \[ \Pr(y_n|\hat{C}_{X_w}) \geq \Pr(y_n|\hat{C}_{\bar{X}_w}), \quad \forall \bar{w} \neq w. \tag{5} \]

  The decoder looks for both the set of $k$ bins $X_w$ as well as the selection of the codeword for each bin, i.e., the selection of codeword index $j_i$ within the $i$th bin, $i \in X_w$, which maximize the conditional probability (5).

- The decoder recovers $\hat{s}$ from $\hat{C}_{X_w} = \{c_{j_i,w}\}_{i \in X_w}$ by setting its $i$th entry, denoted $\hat{s}_i$, to be $\hat{s}_i = q_{j_i}$, for each $i \in X_w\, \, \, \text{and} \, \, \, \hat{s}_i = q_0$, for $i \notin X_w$.

  The ML decoder scans $\binom{n}{k}$ $k^k$ possible subsets of codewords in the codebook, i.e., the $\binom{n}{k}$ possible bins corresponding to indexes which may contain non-zero values, and the $l$ codewords in each such bin. For every scanned subset of codewords, the decoder compares the Boolean OR of each subset which contains $k$ codewords to the quantized register $y_n$. Since the length of each codeword is $b$, the computational complexity is of the order of $O\left(\binom{n}{k}k^kb\right)$ operations.

While the decoding process described above may be computationally complex, it essentially implements a one-to-one mapping from $y_n$ to $\hat{s}$, and can thus be implemented using a standard look-up table. Furthermore, in ongoing work we consider a sub-optimal low-complexity SeQuanS decoder based on the Column Matching (CoMa) method used in [33], [34].

In the following subsection we study the achievable performance, in terms of the tradeoff between quantization rate and distortion, of the proposed SeQuanS system.

D. Achievable Performance

In order to study the achievable performance, we first note that the SeQuanS encoder and decoder are designed to recover the output of the scalar quantizer $Q_{l+1}^1(\cdot)$. Therefore, when the SeQuanS decoder detects the correct set of codewords, the distortion is determined by the scalar quantizer and its resolution, which is dictated by the auxiliary parameter $l$.

To formulate this distortion, define the overall average MSE of the scalar quantizer via

\[ D_n(l) \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[||s_i - Q_{l+1}^1(s_i)||^2\right]. \tag{6} \]

The average MSE (6) is determined by the serial scalar quantizer $Q_{l+1}^1(\cdot)$ and the distribution of $s$. It represents the accuracy of applying $Q_{l+1}^1(\cdot)$ directly to the signal $s$ without using any additional processing, thus operating at quantization rate of $\log(l+1)$ bits per input sample. SeQuanS with rate $R$, which, as we show next, can be much smaller than $\log(l+1)$, is capable of achieving the average MSE when its decoder successfully recovers the correct set of codewords. A sufficient condition for successful recovery in the limit of asymptotically large inputs, and thus for (6) to be achievable, is stated in the following theorem:

**Theorem 1.** The SeQuanS system applied to a sparse signal $s$ with support size $k = O(1)$ achieves the average MSE $D_n(l)$ given in (6) in the limit $n \to \infty$ when the quantization rate $R$ satisfies the following inequality:

\[ R \geq R_\varepsilon(l) \triangleq \max_{1 \leq i \leq k} \left(1 + \varepsilon\right)k \log \left(\frac{n - k}{i \cdot n}\right), \tag{7} \]

for some $\varepsilon > 0$.

**Proof:** The proof is given in the appendix.

Theorem 1 implies that, as $n$ increases, if the number of bits is $b = R \cdot n$ where $R$ satisfies (7), then the average error probability in detecting the SeQuanS codewords approaches zero, decaying exponentially with $n$, and thus the SeQuanS system achieves the average MSE $D_n(l)$ given in (6).

That note the average MSE $D_n(l)$ and the corresponding quantization rate $R_\varepsilon(l)$ both depend on the auxiliary parameter $l$. The dependence of $D_n(l)$ on $l$ is obtained from the quantization mapping used, as well as the distribution of the input $s$. For example, when the entries of $s$ are identically distributed with probability density function (PDF) $f_s(\cdot)$, then, using the Panter-Dite approximation [35], the optimal (non-uniform)
scalar quantizer in the fine quantization regime achieves the following average MSE:
\[ D_n(l) \approx 1 + \frac{1}{2\log(l+1)} \int_{a \in R} f_s^{1/3}(a) \, da. \] (8)

The average MSE in (8) imply that the achievable distortion using scalar quantizers, including conventional architectures such as uniform quantization mappings, can be made arbitrarily small by increasing the resolution log(l + 1).

While the average MSE directly depends on the quantization mapping, the quantization rate \( R_e(l) \) is invariant to the setting of \( Q_{l+1}^s(\cdot) \), and is obtained as the maximal value of the right hand side of (7). To avoid the need to search for the maximal value in (7), we state an upper bound on \( R_e(l) \) in the following corollary:

**Corollary 1.** The quantization rate \( R_e(l) \) in (7) is upper bounded by
\[ R_e(l) \leq (1 + \epsilon) \frac{l}{n} \log(n \cdot l). \] (9)

**Proof:** The corollary is obtained by substituting in (7) the upper bound \( \log\binom{n+k}{n-k} \leq l \log n \) using Stirling’s approximation [26], [33].

We note that when \( k = O(1) \), the upper bound (9) tends to zero for any fixed \( l \) as \( n \) grows. Consequently, for large \( n \), SeQuanS requires significantly smaller quantization rates to achieve \( D_n(l) \) compared to directly applying \( Q_{l+1}^s(\cdot) \) to \( s \), which requires a rate of \( \log(l+1) \) to achieve the same average MSE. This gain, which demonstrates the ability of SeQuanS to exploit the underlying sparsity of \( s \), is also observed in the simulations study presented in Section IV.

Corollary 1 can be used to determine the quantization rate for achieving a desirable MSE for a given family of serial scalar quantization mappings: The auxiliary parameter \( l \) is set to the minimal value for which \( D_n(l) \) is not larger than the desirable distortion. Next, using the resulting \( l \), the quantization rate can be obtained using the right-hand side of (9). Theorem 1 guarantees that, for large input size \( n \), the desirable distortion is achievable when using SeQuanS with the selected quantization rate. In fact, in the numerical study presented in Section IV we demonstrate that, by properly tuning \( l \), the proposed system can achieve substantial MSE gains over previously proposed approaches for quantizing sparse signals.

The bound on the quantization rate required to approach \( D_n(l) \) given in Corollary 1 can also be used to characterize the asymptotic growth rate of the number of quantization bits used by the SeQuanS system, \( b \), as stated in the following corollary:

**Corollary 2.** The MSE \( D_n(l) \) can be approached as \( n \) increases when the number of quantization bits \( b \) grows as
\[ b = O(k \log n + k \log l). \] (10)

Corollary 2 implies that, besides the obvious linear dependence in \( k \log l \), the required number of bits grows proportionally to a logarithmic factor of \( n \), which depends on the sparsity pattern size \( k \). A similar asymptotic growth in the number of bits, i.e., proportional to \( k \log n \), was also shown to be sufficient to achieve a given distortion when using CS-based methods in [17, Thm. 2]. However, our numerical study presented in Section IV demonstrates that despite the similarity in the asymptotic growth of the number of bits, when \( b \) is fixed, SeQuanS achieves improved reconstruction accuracy compared to CS-based techniques.

Substituting (10) in the complexity analysis in Subsection III-D, allows us to characterize the computational burden of the ML decoder utilized in the SeQuanS scheme, in the following corollary:

**Corollary 3.** The SeQuanS scheme with the ML decoder detailed in Subsection III-C is capable of achieving the MSE \( D_n(l) \) in the limit \( n \to \infty \) with a computational complexity on the order of \( O\left(\left(\frac{n}{l}\right)^{18}k^2 \log n + \left(\frac{n}{l}\right)^{18}k^2 \log l\right) \) operations.

The complexity of the SeQuanS decoder is significantly affected by the size of the sparsity pattern \( k \) in a much more dominant manner compared to the size of the input signal \( n \), and the resolution of the scalar quantizer \( l \). While this implies that the SeQuanS system is most computationally efficient for highly sparse inputs, the proposed mechanism is applicable for any size of the sparsity pattern.

**E. Discussion**

We next discuss the practical aspects of this method and its rationale. In particular, we first discuss the benefits which stem from the SeQuanS architecture and compare it to related schemes for quantizing sparse signals, such as direct application of scalar quantizers as well as compress-and-quantize [17]–[20], [23]. Then, we elaborate on the relationship of SeQuanS with group testing theory.

1) **Practical benefits and comparison with related schemes:** The SeQuanS system is specifically designed to utilize scalar ADCs in a serial manner. The resulting structure can be therefore naturally implemented using practical hardware-limited serial scalar ADC architectures [3]. Moreover, SeQuanS is tailored to exploit an underlying sparsity of the input signal. Straight-forward application of a serial scalar ADC requires \( n \cdot \log(l+1) \) bits to achieve the distortion \( D_n(l) \) in (6). Our proposed SeQuanS, which exploits the sparsity of the input by further encoding the ADC output in a serial manner, requires \( b = O(k \log(nl)) \) bits to achieve the same MSE, as follows from Corollary 2. This implies that for sparse signals, i.e., when \( k \ll n \), SeQuanS can significantly reduce the number of bits while utilizing serial-scalar ADCs for acquisition, by introducing an additional encoding applied in a serial manner at its output. The resulting approach thus bears some similarity to previously proposed universal quantization methods which are based on applying entropy coding to the output of a quantizer. See [36] for scalar quantizers and [37] for vector quantizers. Indeed, since the codewords representing the quantized value are generated according to a Bernoulli distribution with mean value \( \frac{\log(2)}{\log(k+1)} \) and the outcome \( y_n \) is the Boolean OR of \( k \) inputs, it can be shown that its entries approach being independent and equally distributed on the...
set \{0,1\} for large values of \(k\), namely, the optimal lossless encoded representation, as achieved using entropy coding [26, Ch. 5]. Nonetheless, to apply conventional entropy coding, one must first quantize all the entries of the input (or at least a large block of input entries) before applying the encoding process, requiring a large number of bits to store and represent this quantized block. SeQuanS, which is specifically designed to operate in a serial manner, updates the same \(b\)-bits register on each incoming samples, thus avoiding the need to store the output of the serial scalar ADC \(Q_{l+1}^i()\) prior to its encoding.

Arguably the most common approach considered in the literature for quantization of sparse signals is based on CS techniques. In these methods, a sensing matrix is used to linearly combine the sparse signal into a lower-dimensional vector, which is then quantized, either using optimal vector quantization, as in [21], or more commonly, via some scalar continuous-to-discrete mapping, as in [17]–[20]. When the input signal is a time sequence acquired in a sequential manner, as considered in our problem formulation, such CS based techniques need to store the incoming samples in the analog domain prior to their combining using the sensing matrix\(^2\). This requirement, which does not exist for our proposed SeQuanS, limits the applicability of these proposed schemes, especially for large-dimensional inputs, i.e., in the regime typically considered in the literature.

An additional benefit of the proposed SeQuanS compared to CS-based methods, stems from our usage of binary codebooks for compression. By using binary codes originating from group testing theory, we are able to achieve improved immunity to measurement errors compared to operating over fields of higher cardinality. This benefit is translated to more accurate digital representations, as demonstrated in our numerical results in Section IV.

A possible drawback of SeQuanS compared to CS-schemes stems from the fact that SeQuanS is designed assuming that the signal \(s\) is sparse, i.e., that at most \(\ell\) of its entries are non-zero. CS methods are commonly capable of reliably recovering signals which are sparse in an alternative domain, namely, when there exists a non-singular matrix \(P\) such that \(Ps\) is sparse. It is noted though that SeQuanS can still be applied to such signals by first projecting the signal using the matrix \(P\), resulting in a sparse signal which can be represented using SeQuanS. Such application however requires the entire signal to be first acquired, as is the case with conventional CS methods.

Finally, we note that while CS based techniques typically require the ratio between the sparsity pattern size \(k\) and the input dimensionality \(n\) to be upper bounded, our proposed SeQuanS can be applied for any ratio between \(k\) and \(n\).

2) Relationship to group testing theory: As mentioned in Subsection III-A, the SeQuanS code construction is inspired by codebooks designed for the group testing problem. Group testing first originated from the need to identify a small subset \(k\) of infected draftees with syphilis from a large set of \(n\) candidates. Our code construction overcomes this difference by exploiting recent code designs targeting extended group testing models, and in particular, those considered in [30] and in the secure group testing framework [32].

The resulting group testing based code design leads to a compact and accurate digital representation. In particular, due to the binning structure of the code suggested, when \(k\) inputs are different from zero there are only \(\binom{n}{k}\) possible subsets of codewords from which the output of the encoder is selected, and not \(\binom{n}{\ell}\) considering the naive codebook which assigns a different codeword to each quantized input value without binning. This significantly reduces the number of bits required in the outcome vector.

Finally, we note that the construction of the suggested code does not depend on the distribution of the input signal, which is similar to universal quantization methods [36]. In fact, the distribution of \(s\) only affects the MSE induced by the serial scalar quantizer \(Q^l_i()\). The codebook presented in Subsection III-A is designed to allow reliable reconstruction under the worst case scenario, i.e., the setting in which \(\{Q^l_i(s_i)\}_{i=1}^n\) are i.i.d. uniformly distributed. Intuitively, the quantization rate required to achieve the MSE \(D_n(l)\) can be further reduced by exploiting a-priori information on the input distribution. This approach was considered for the original group testing problem with, e.g., Poisson priors in [38]. We leave investigation of this approach under our quantization framework for future study.

IV. NUMERICAL EVALUATIONS

Here, we evaluate the achievable distortion of the proposed SeQuanS scheme in a simulations study, for a fixed and finite signal size \(n\). To that aim, we consider two sparse sources with sizes \(n \in \{100,50\}\) and support sizes \(k \in \{3,2\}\), respectively. To generate each signal, we randomly select \(k\) indexes, denoted \(\{i_j\}_{j=1}^\ell\), and then choose the values of \(s\) on these indexes to be i.i.d. zero-mean unit-variance Gaussian random variables (RVs), while the remaining entries are set to zero.

Each of the generated signals is quantized and represented in digital form using each of the following methods:

- **SeQuanS system with** \(l = \left\lceil \frac{\ln 2 \times 2 - \epsilon}{\ln 2} \right\rceil\) following (9), where \(\epsilon\) is selected in the range \(\epsilon \in [0.8,1.3]\). Here, the continuous-to-discrete mapping \(Q_{l+1}^i()\) implements uniform quantization over the region \([-2,2]\).
- A uniform scalar quantizer with support \([-2,2]\) applied to each entry of \(s\), mapping each decision region to its centroid. This system, which models the direct application of a serial scalar ADC to the sparse signal \(s\) can be

\(^2\)One may also store only the lower-dimension compressed vector and update its entries on each incoming input sample. Yet, this approach still requires the storage of a large amount of samples in analog as quantization can only be carried out once the complete signal is compressed.
applied only when the quantization rate satisfies $R \geq 1$ to utilize quantizers with at least one bit.

- A compress-and-quantize system which first compresses $s$ into $\mathbb{R}^m$, where $m$ is selected in the range $[6k; 20k]$ to minimize the MSE. The compression is carried out using a sensing matrix $A \in \mathbb{R}^{m \times n}$ whose entries are i.i.d. zero-mean unit variance Gaussian RVs. The compressed signal $As$ is quantized using a uniform scalar quantizer with support $[-2, 2]$. The digital representation $\hat{s}$ is then recovered using the quantized iterative hard thresholding (QIHT) method [39] as well as fast iterative soft thresholding (FISTA) [40].

All of the above schemes are compared with the same number of bits $b = R \cdot n$, and the MSE is computed by averaging the squared error over 100 Monte Carlo simulations.

The empirically evaluated MSEs of the considered quantization systems versus the quantization rate $R$ are depicted in Figs. 4-5 for the setups with $(n, k) = (100, 3)$ and $(n, k) = (50, 2)$, respectively. Observing Figs. 4-5, it is noted that the proposed SeQuanS system achieves superior representation accuracy and that its resulting MSE is not larger than $10^{-4}$ for quantization rates $R \geq 0.8$. For comparison, directly applying a scalar quantizer to the sparse signal is feasible only for $R \geq 1$, and its achievable MSE is only slightly less than 1. This degraded performance of directly applying scalar quantizers stems from the fact that for the considered rates $R \in [1, 2)$, this quantization mapping implements a one-bit sign quantization of the entries of $s$. Since most of the entries of $s$ are zero, this quantization rule induces substantial distortion.

The MSE performance of the CS-based quantization scheme improves much less dramatically with the quantization rate $R$ compared to SeQuanS. For example, for the scenario depicted in Fig. 4, the SeQuanS system achieves MSE of $2 \cdot 10^{-3}$ for $R = 0.5$, while the CS-based systems achieve an MSEs of $1.2 \cdot 10^{-3}$ and $1.4 \cdot 10^{-2}$ for the QIHT and FISTA decoders, respectively, i.e., a gap of approximately 7 dB. However, for quantization rate of $R = 1$, the corresponding MSE values are $1.7 \cdot 10^{-6}$, $2 \cdot 10^{-3}$, and $9 \cdot 10^{-3}$, for the SeQuanS system, CS with QIHT recovery, and CS with FISTA recovery, respectively, namely, performance gaps of $30 - 37$ dB in MSE. For all considered scenarios, the QIHT recovery scheme, which is specifically designed for reconstructing sparse signals from compressed and quantized measurements, outperforms the FISTA method which considers general sparse recovery.

The results presented in this section demonstrate the potential of SeQuanS as a quantization scheme for sparse signals which is both accurate as well as suitable for implementation with conventional serial scalar ADCs.

V. CONCLUSIONS

In this paper we proposed SeQuanS, a quantization system designed for representing sparse signals acquired in a sequential manner. SeQuanS combines code structures from group testing theory with the limitations and characteristics of conventional ADCs. We derived the achievable MSE of the proposed scheme in the asymptotic signal size regime and characterized its complexity. Our simulation study demonstrates the substantial performance gain of SeQuanS compared to directly applying a serial scalar ADC, as well as to CS-based methods.

APPENDIX

To prove the Theorem 1, we first provide a reliability bound which guarantees accurate reconstruction of the quantized representation $\{Q_{l+1}^i(s_i)\}_{i=1}^n$ from $y_n$. Then, we show that this bound results in the condition on the quantization rate stated in Theorem 1. An achievability bound on the required number of bits is stated in the following lemma:

**Lemma 1.** If for some $\varepsilon > 0$ independent of $n$ and $k$, the number of bits used for digital representation satisfies

$$b \geq \max_{1 \leq i \leq k} \frac{(1+\varepsilon)k}{i} \log \left(\frac{n-k}{i}\right)^i,$$

then, under the code construction of Section III, as $n \to \infty$ the average error probability to recover $\{Q_{l+1}^i(s_i)\}_{i=1}^n$, given by $\frac{1}{n} \sum_{i=1}^n \Pr(\hat{s}_i \neq Q_{l+1}^i(s_i))$, approaches zero exponentially.

**Proof.** The Lemma follows from [30, Lemma 1], whose proof is based on [32, Lemma 2]. \(\square\)
We note that the corresponding bound in [41, Theorem III.1], which studied group testing over a binary field, can be considered as a special case of Lemma 1 with \( l = 1 \), i.e., using one bit quantizers. In particular, since we consider quantizers with arbitrary resolution, the bound in Lemma 1 must account for the fact that the codewords have to be selected from different bins, as \( l \) can be larger than one.

Now, Lemma 1 yields a sufficient condition for the digital representation \( \hat{s} \) to approach the directly quantized \( s \), for which the MSE is \( D_n(l) \) given in (6). Since the quantization rate is given by \( R = \frac{k}{n} \), the condition (11) becomes

\[
R \geq \max_{1 \leq k \leq n} \left( \frac{1 + \epsilon}{1 - n} \right) \log \left( \binom{n}{i} \cdot l \right),
\]

proving the theorem.

REFERENCES